Symposium for Gordon Plotkin

ANALYTIC FUNCTORS & DOMAIN THEORY

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*with thanks to Martin Hyland

MOTIVATIONS FOR THE TALK

Points of contact with Gordon

- Domain theory
  - An area of fundamental contributions by Gordon
  - Our collaboration on AxDT

Rich model of combinatorial, computational, logical, and physical structures

- Combinatorial species, differential calculus, linear logic, domain theory, λ-calculus, quantum structures, ...

ANALYTIC FUNCTIONS

- Admit a Taylor-series development:

$$f(x) = \sum_{n \in \mathbb{N}} a_n \frac{x^n}{n!}$$

- Are infinitely differentiable:

$$a_n = f^{(n)}(0)$$ — series of Taylor coefficients

- Extensional counterparts of convergent formal exponential power series

power series $\sum_n a_n x^n$

exponential power series $\sum_n a_n x^n/n!$

- Example:

$$e^x = \sum_n x^n/n!$$

GENERATING FUNCTIONS

$$f(x) = \sum_n a_n \, x^n/n!$$

number of combinatorial structures of a certain type (e.g. trees, cycles, etc.) on a set of n tokens

GENERATING FUNCTORS

[Joyal]

$$F(X) = \sum_n (A_n \times X^n)/\mathfrak{S}_n$$

- A_n = set of structures of type A on n tokens

  May be permuted in the structure; hence a type A is given by symmetric group actions
  $$A_n \times \mathfrak{S}_n \to A_n$$

- X^n = labellings/colourings of the tokens in X

- $(A_n \times X^n)/\mathfrak{S}_n$
  $$(a \cdot \sigma, \ell) \sim (a, \sigma ; \ell)$$

- Example:
  $$e^X = \sum_n X^n/\mathfrak{S}_n = \text{finite multisets on } X$$

SPECIES OF STRUCTURES & ANALYTIC FUNCTORS

[Joyal]

A type of structure

$A = \{ A_n \times \mathfrak{S}_n \to A_n \}_n$

formal exponential power series

$\tilde{A}(X) = \sum_n (A_n \times X^n)/\mathfrak{S}_n$

exponential generating functors

intensional $\leftarrow$ $\boxed{\text{two views}}$ $\rightarrow$ extensional

$A : \mathbf{Bij} \to \mathbf{Set}$ $\longmapsto$ $\tilde{A} : \mathbf{Set} \to \mathbf{Set}$

finite sets & bijections

the analytic functor associated to the species A

Def: $F : \mathbf{Set} \to \mathbf{Set}$ is analytic iff$_{\text{def}}$ $\exists$ a species A such that $\tilde{A} \cong F$

▶ A is the Taylor series of F

MULTIVARIABLE CASE

$$\frac{\mathrm{Set}^k \xrightarrow{F} \mathrm{Set}^\ell}{\left\{\mathrm{Set}^k \xrightarrow{F_i} \mathrm{Set}\right\}_{i=1,\dots,\ell}}$$

- Analytic functors $\mathrm{Set}^k \to \mathrm{Set}$:

$$\tilde{A}(X) = \sum_{n_1,\dots,n_k} \left(A_{n_1,\dots,n_k} \times \prod_i X_i^{n_i} \right) \Big/ \prod_i \mathfrak{S}_{n_i}$$

with A a k-sorted species :

$$A = \left\{ A_{n_1,\dots,n_k} \times \prod_i \mathfrak{S}_{n_i} \to A_{n_1,\dots,n_k} \right\}$$

- We proceed to generalise and systematise the concept.

ANALYTIC FUNCTORS BETWEEN PRESHEAF CATEGORIES

$$\widehat{\mathbb{K}} \longrightarrow \widehat{\mathbb{L}}$$

- Presheaf categories: $\widehat{\mathbb{X}} =^{def} \mathbf{Set}^{\mathbb{X}^{op}}$

  Two views:

  - from domain theory: prime algebraic lattice
  - from linear algebra: vector/Hilbert space

- Basis embedding:

$$\mathbb{X} \hookrightarrow \widehat{\mathbb{X}}$$

$$x \longmapsto \vec{x}$$

where $(\vec{x})_{x'} =^{def} \mathbb{X}(x', x)$

Cayley
Dedekind
Yoneda
Grothendieck

- Every presheaf $X \in \widehat{\mathbb{X}}$ is a linear combination of the basis vectors $\vec{x}$ $(x \in \mathbb{X})$:

$$X \cong \int^{x \in \mathbb{X}} \vec{x} \cdot X_x$$

$$= \left(\sum_{x \in \mathbb{X}} \vec{x} \cdot X_x \right) \Big/ \sim$$

FOCK SPACE ON CATEGORIES

mathematical model of quantum systems of many identical, non-interacting particles

$$!\mathbb{X} = \sum_n \mathbb{X}^n /\!\!/ \mathfrak{S}_n = e^{\mathbb{X}}$$

where

$$\mathbb{X}^n /\!\!/ \mathfrak{S}_n$$

has <u>objects</u>: $(x_1, \dots, x_n)$

<u>morphisms</u>: $(x_i)_i \xrightarrow[(\sigma, f)]{} (y_j)_j$

with $\sigma \in \mathfrak{S}_n$

$f = \{ f_i : x_i \to y_{\sigma i} \}_i$

<u>NB.</u> $e = !\mathbb{1} \simeq \mathbf{Bij}$

$e^{\mathbb{0}} \simeq \mathbb{1}$

$e^{\mathbb{X}+\mathbb{Y}} \simeq e^{\mathbb{X}} \times e^{\mathbb{Y}}$

$$\begin{array}{c} \mathbf{SymmMonCat} \\ \wr\!\wr \\ \mathbf{CommMon(Cat)} \\ \uparrow \dashv \downarrow \\ e^{(-)} \circlearrowleft \ \mathbf{Cat} \end{array}$$

GENERALISED

SPECIES & ANALYTIC FUNCTORS

$$!\mathbb{1} \cong \mathrm{Bij} \to \mathrm{Set} \qquad \mapsto \qquad \mathrm{Set} \to \mathrm{Set}$$

$$![k] \simeq \mathrm{Bij}^k \to \mathrm{Set} \qquad \mapsto \qquad \mathrm{Set}^k \to \mathrm{Set}$$

generalise to

$$!\mathbb{K} \xrightarrow[A]{} \widehat{\mathbb{L}} \qquad \mapsto \qquad \widehat{\mathbb{K}} \xrightarrow[\widetilde{A}]{} \widehat{\mathbb{L}}$$

$$\widetilde{A}(X)_\ell =^{\mathrm{df}} \Big(\sum_{K \in !\mathbb{K}} \underbrace{A(K)_\ell}_{\text{coefficients}} \times \underbrace{\Big(\prod_{k \in K} X_k \Big)}_{\text{monomials}} \Big) \Big/ \sim$$

$$= \int^{K \in !\mathbb{K}} A(K)_\ell \times \Big(\prod_{k \in K} X_k \Big)$$

A CLASS OF EXAMPLES

Def: For $\mathbb{K}$ and $\mathbb{L}$ sets,

$$F : \mathbf{Set}^{\mathbb{K}} \to \mathbf{Set}^{\mathbb{L}}$$

have power series expansion

is normal iff$_{\mathrm{def}}$ [Girard]

$$F(X)_\ell \cong \sum_{\underbrace{M \in \mathcal{M}(\mathbb{K})}_{\text{finite multisets (= monomials) on } \mathbb{K}}} A(M)_\ell \cdot \Big(\prod_{k \in M} X_k \Big)$$

for some $A : \mathcal{M}(\mathbb{K}) \to \mathbf{Set}^{\mathbb{L}}$

Prop: Every normal functor is analytic.

Ex: The free monoid endofunctor on Set,

$$F(X) = \sum_n X^n ,$$

is analytic.

GENERAL ABSTRACT VIEW

- Superposition functor

$$\begin{tikzcd} \mathbb{X} \arrow[r, hook] \arrow[dr, "\vec{(\cdot)}"'] & !\mathbb{X} \arrow[d, dashed, "S"] \arrow[dl, phantom, "\cong"] \\ & \widehat{\mathbb{X}} \end{tikzcd}$$

basis embedding $\rightsquigarrow \vec{(\cdot)}$

$$\begin{tikzcd} (x_1, \ldots, x_n) \arrow[d, mapsto] \\ \sum_i \vec{x_i} \end{tikzcd}$$

- Analytic functors

$$\begin{tikzcd} !\mathbb{K} \arrow[r, "S"] \arrow[dr, "A"'] & \widehat{\mathbb{K}} \arrow[d, "\widetilde{A}"] \arrow[dl, phantom, "\overset{\mathrm{Lan}}{\Rightarrow}"] \\ & \widehat{\mathbb{L}} \end{tikzcd}$$

A ~ coefficients

$\widetilde{A}$ ~ least universal extension of A along S

Analytic Functors

~ Basic Theory ~

Def: $F : \widehat{\mathbb{K}} \to \widehat{\mathbb{L}}$ is analytic

iff$_{\text{def}}$ $\exists\, A : {!}\mathbb{K} \to \widehat{\mathbb{L}}$ s.t. $\widetilde{A} \cong F$

the Taylor series of F

Prop:

$A \cong B : {!}\mathbb{K} \to \widehat{\mathbb{L}}$

iff

$\widetilde{A} \cong \widetilde{B} : \widehat{\mathbb{K}} \to \widehat{\mathbb{L}}$

Hence the concept of Taylor series is well defined up to isomorphism

ANALYTIC FUNCTORS

~ BASIC THEORY ~

Prop: Identities are analytic

$$I_{\mathbb{K}}(K)_k = !\mathbb{K}[(k), K]$$

Thm: Analytic functors are closed under composition

$$\widetilde{B}\ \widetilde{A} \cong \widetilde{B \circ A}$$

[FGHW] preprint

$\circ$: substitution tensor product

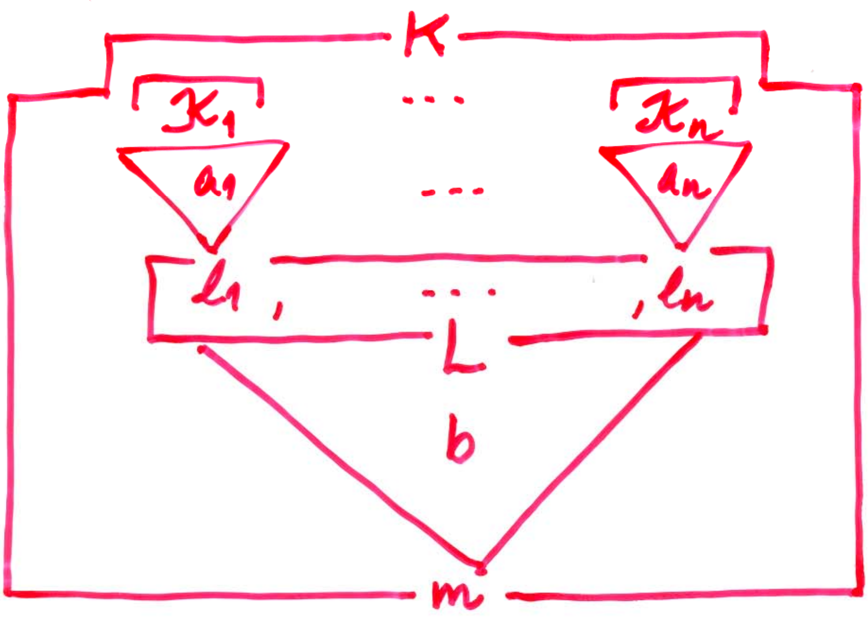
THE SUBSTITUTION TENSOR PRODUCT

[Kelly]
[MFPS 2006]

$$\frac{A : !\mathbb{K} \to \widehat{\mathbb{L}} \qquad B : !\mathbb{L} \to \widehat{\mathbb{M}}}{B \circ A : !\mathbb{K} \to \widehat{\mathbb{M}}}$$

$$(B \circ A)(K)_m = \int^{L \in !\mathbb{L}} B(L)_m \times A^*(K)_L$$

$$A^*(K)_L = \int^{x \in \mathbb{K}^{|L|}} \prod_{\ell \in L} A(x_\ell)_\ell \times \Big[\bigoplus_{\ell \in L} x_\ell , K\Big]$$



$a_i \in A(x_i)_{\ell_i}$

$b \in B(L)_m$

$\in (B \circ A)(K)_m$

type of substitutions of A's for B's

as in linear algebra

LINEAR FUNCTORS

~ a class of analytic functors ~

- Arise from <u>matrices</u>:

$$\frac{\mathbb{L}^{op} \times \mathbb{K} \to \mathbf{Set}}{\mathbb{K} \xrightarrow[M]{} \widehat{\mathbb{L}}}$$

- The <u>linear functor</u> associated to M is

$$\overline{M} : \widehat{\mathbb{K}} \longrightarrow \widehat{\mathbb{L}}$$

given by

$$\overline{M}(X)_\ell = \int^{k \in \mathbb{K}} M(k)_\ell \cdot X_k$$

<u>N.B.</u> Linear functors are analytic:

$$\overline{M} \cong \widetilde{M^{\#}}$$

where

$$M^{\#}(K)_\ell = \int^{k \in \mathbb{K}} M(k)_\ell \cdot [\,(k), K\,]$$

Linear Functors
~ Basic Theory ~

Def: $F : \hat{\mathbb{K}} \to \hat{\mathbb{L}}$ is linear

iff $\exists\, M : \mathbb{K} \to \hat{\mathbb{L}}\ .\quad F \cong \overline{M}$

the matrix of F

N.B. $\overline{M} \cong \overline{N}$ iff $M \cong N$

Abstract view:

$$\begin{tikzcd} \mathbb{K} \arrow[r, hook, "\vec{(\cdot)}"] \arrow[dr, "M"'] & \hat{\mathbb{K}} \arrow[d, "\overline{M}"] \arrow[dl, phantom, "\overset{\mathrm{Lan}}{\cong}"] \\ & \hat{\mathbb{L}} \end{tikzcd}$$

Prop: Identities are linear

$$I_{\mathbb{K}}(k)_{k'} = \mathbb{K}[k', k]$$

Prop: Linear functors are closed under composition

$$\overline{N}\ \overline{M} \cong \overline{N \cdot M}$$

matrix multiplication

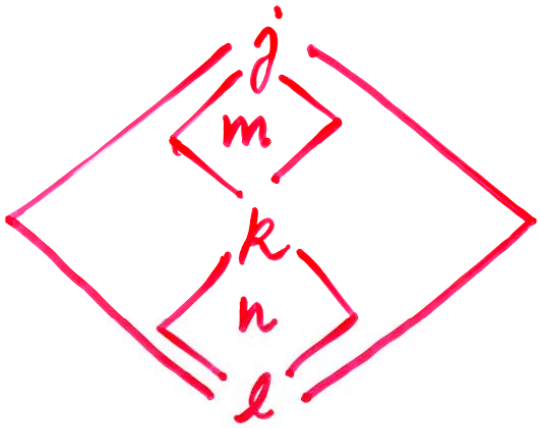
MATRIX MULTIPLICATION

$$\frac{M : \mathbb{J} \to \widehat{\mathbb{K}} \qquad N : \mathbb{K} \to \widehat{\mathbb{L}}}{N \cdot M : \mathbb{J} \to \widehat{\mathbb{L}}}$$

$$(N \cdot M)(j)_l = \int^{k \in \mathbb{K}} N(k)_l \times M(j)_k$$

$m \in M(j)_k$

$n \in N(k)_l$



$\in (N \cdot M)(j)_l$

LINEAR FUNCTORS
~ BASIC THEORY ~

Thm: A functor is linear iff it preserves colimits

$$F(X)_\ell \cong F\left(\int^{k \in \mathbb{K}} \vec{k} \cdot X_k\right)_\ell$$

$$\cong \int^{k \in \mathbb{K}} \underbrace{F(\vec{k})_\ell}_{\text{the matrix of } F} \cdot X_k$$

THE LANDSCAPE

substitution
tensor product

Esp

$$
\begin{tikzcd}
!\mathrm{GCat} \arrow[r, "\widehat{(\cdot)}"] \arrow[d, shift right=2] & \mathrm{CAT} \\
\mathrm{CommMon(Cat)} \arrow[u, shift right=2] \arrow[u, phantom, "\dashv"] \arrow[dr, phantom, "\supset"] & \\
& \mathrm{SymmMonCat}
\end{tikzcd}
$$

THE LANDSCAPE

substitution
tensor product

$$\begin{tikzcd} \mathbf{Esp} \\ \mathbf{Mat} \simeq \mathbf{Lin} \arrow[u, "(\cdot)^\#"'] \end{tikzcd}$$

compact
closed
bicategory
[DS]

$$\begin{tikzcd} !\,\mathcal{G}\,\mathbf{Cat} \arrow[r, "\widehat{(\cdot)}"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & \mathbf{CAT} \\ \mathbf{CommMon(Cat)} \arrow[u, shift right] & \end{tikzcd}$$

$\mathbf{CommMon(Cat)} \subsetneq \mathbf{SymmMonCat}$

$$\mathbf{Mat}(\mathbb{K},\mathbb{L}) = [\mathbb{K},\mathbb{L}^\wedge] \cong \mathbf{CoCont}[\widehat{\mathbb{K}},\mathbb{L}^\wedge] \cong \mathbf{Lin}(\widehat{\mathbb{K}},\mathbb{L}^\wedge)$$

THE LANDSCAPE

substitution
tensor product

$$
\begin{tikzcd}
\mathbf{Esp} & & \\
\mathbf{Mat} \simeq \mathbf{Lin} \arrow[u, "(-)^{\#}"] \arrow[dr, phantom, "\wr"] & & \\
& \underline{\mathbf{Kl}}(\hat{\cdot}) \arrow[loop above, "!\ \text{monad lifting}"] \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] & \\
& \text{dist. law} \atop !\circlearrowleft\mathbf{Cat} \arrow[u, shift left=2] \arrow[r, "\widehat{(\cdot)}"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & \mathbf{CAT} \\
& \mathbf{CommMon}(\mathbf{Cat}) \arrow[u, shift right=2] \arrow[dr, phantom, "\wr"] & \\
& & \mathbf{SymmMonCat}
\end{tikzcd}
$$

compact
closed
bicategory
[DS]

Kleisli structure [FGHW]

$$\mathrm{Mat}(\mathbb{K},\mathbb{L}) = [\mathbb{K},\hat{\mathbb{L}}] \cong \mathrm{CoCont}[\hat{\mathbb{K}},\hat{\mathbb{L}}] \cong \mathrm{Lin}(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

THE LANDSCAPE

substitution tensor product

Cartesian Closure

[FGHW] preprint

linear ! comonad

compact closed bicategory [DS]

! monad lifting

dist. law

Kleisli structure [FGHW]

$$
\begin{tikzcd}
\mathrm{CoKl}(!) \arrow[r, phantom, "\simeq"] \arrow[d, shift left, ""] \arrow[d, phantom, "\dashv"] & \mathrm{Esp} & & \\
\mathrm{Mat} \arrow[loop left, "!"] \arrow[u, shift left] \arrow[ur, "(\cdot)^{\#}"'] \arrow[r, phantom, "\simeq"] & \mathrm{Lin} \arrow[dr, phantom, "\simeq"] & & \\
& & \underline{\mathrm{Kl}}(\hat{\cdot}) \arrow[loop above, "!"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\
& & \mathrm{Cat} \arrow[loop left, "!"] \arrow[u, shift left] \arrow[r, "\widehat{(\cdot)}"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & \mathrm{CAT} \\
& & \mathrm{CommMon}(\mathrm{Cat}) \arrow[u, shift right] \arrow[d, phantom, "\simeq"] & \\
& & \mathrm{SymmMonCat} &
\end{tikzcd}
$$

$$\mathrm{Mat}(\mathbb{K},\mathbb{L}) = [\mathbb{K},\hat{\mathbb{L}}] \cong \mathrm{CoCont}[\hat{\mathbb{K}},\hat{\mathbb{L}}] \cong \mathrm{Lin}(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

THE LANDSCAPE

substitution : tensor product

Cartesian Closure

[FGHW] preprint

Differential & Quantum structure [FOSSACS 05, CKC 06]

$$
\begin{tikzcd}
\mathrm{CoKl}(!) \arrow[r, phantom, "\simeq"] \arrow[dr, shift left] \arrow[dr, phantom, "\dashv"] & \mathrm{Esp} & & & \\
 & \mathrm{Mat} \arrow[loop left, "!"] \arrow[u, "(-)^{\#}"] \arrow[ul, shift left] \arrow[r, phantom, "\simeq"] & \mathrm{Lin} \arrow[dr, phantom, "\simeq"] & & \\
 & & & \underline{\mathrm{Kl}}(\hat{-}) \arrow[loop above, "!"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\
\underline{\mathrm{Maps}}(\mathrm{Mat}) \arrow[uur] \arrow[rrr, phantom, "\simeq"] & & & {!\mathrm{Cat}} \arrow[u, shift left] \arrow[r, "\widehat{(-)}"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & \mathrm{CAT} \\
 & & & \mathrm{CommMon}(\mathrm{Cat}) \arrow[u, shift right] \arrow[dr, phantom, "\simeq"] & \\
 & & & & \mathrm{SymmMonCat}
\end{tikzcd}
$$

linear ! comonad

compact closed bicategory [DS]

! monad lifting

bicategorical lim/colim coincidence [CF]

dist. law

Kleisli structure [FGHW]

$$\mathrm{Mat}(\mathbb{K},\mathbb{L}) = [\mathbb{K},\hat{\mathbb{L}}] \cong \mathrm{CoCont}[\hat{\mathbb{K}},\hat{\mathbb{L}}] \cong \mathrm{Lin}(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

Maps = left adjoints $\rightsquigarrow$ $\left(\text{iff } \begin{array}{l} RR^{o} \leq \mathrm{id} \ \& \ R^{o}R \geq \mathrm{id} \\ R \text{ is functional} \end{array} \right)$

↝ vs. CALCULUS [FOSSACS 2005]

THE ALGEBRA OF SPECIES

~ MULTIPLICATION ~

(by Day's tensor product)

$$\frac{F, G : !\mathbb{K} \nrightarrow \mathbb{L}}{F \bullet G : !\mathbb{K} \nrightarrow \mathbb{L}} \qquad \begin{array}{l} \text{in } \mathit{Mat} \\ \text{in } \mathit{Mat} \end{array}$$

$$\begin{tikzcd}
\text{in } \mathit{Cat} & !\mathbb{K} & !\mathbb{K} \times !\mathbb{K} \arrow[l, "\oplus"'] & \mathbb{L} \times \mathbb{L} & \mathbb{L} \arrow[l, "\Delta"'] & \times \arrow[d, mapsto] \\
\text{in } \mathit{Mat} & !\mathbb{K} \arrow[r, "\shortmid"{marking}, "\oplus^r"'] \arrow[rrr, bend right=40, "\mathrm{def}", "F \bullet G"'] & !\mathbb{K} \times !\mathbb{K} \arrow[r, "\shortmid"{marking}, "F \otimes G"'] & \mathbb{L} \times \mathbb{L} \arrow[r, "\shortmid"{marking}, "\Delta^r"'] & \mathbb{L} & \otimes
\end{tikzcd}$$

CONVOLUTION

cf. [Sweedler]

THE ALGEBRA OF SPECIES

~ DIFFERENTIATION ~

- Linear structure of Mat:

$$\underline{\mathrm{lin}}(\mathbb{K},\mathbb{L}) = \mathbb{K}^{\circ} \times \mathbb{L}$$

- Closed structure of Esp:

$$\underline{\mathrm{hom}}(\mathbb{K},\mathbb{L}) = !\mathbb{K}^{\circ} \times \mathbb{L}$$

- Higher-order differential structure:

$$\underline{\mathrm{hom}}(\mathbb{K},\mathbb{L}) \longrightarrow \underline{\mathrm{hom}}(\mathbb{K}, \underline{\mathrm{lin}}(\mathbb{K},\mathbb{L}))$$

The displacement action homomorphism

$$!\mathbb{K}^{\circ} \xleftarrow{\delta} \mathbb{K}^{\circ} \times !\mathbb{K}^{\circ} \qquad \text{in Cat}$$

yields

$$!\mathbb{K}^{\circ} \times \mathbb{L} \xrightarrow{\delta^{r} \otimes I_{\mathbb{L}}} \mathbb{K}^{\circ} \times !\mathbb{K}^{\circ} \times \mathbb{L}$$

Sample Algebraic Laws

cf. [Ehrhard & Regnier]
[Blute & Cockett & Seely]

<u>Bialgebra laws</u>:

The canonical 2-cells

$$
\begin{tikzcd}
 & !\mathbb{A} \otimes !\mathbb{A} \arrow[r, "\oplus"] \arrow[d, "\oplus^{r} \otimes \oplus^{r}"'] & !\mathbb{A} \arrow[r, "\oplus^{r}"] \arrow[d, phantom, "\Uparrow"] & !\mathbb{A} \otimes !\mathbb{A} & \\
 & !\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{A} \arrow[rr, "\cong", "\mathrm{I} \otimes \sigma \otimes \mathrm{I}"'] & & !\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{A} \arrow[u, "\oplus \otimes \oplus"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{I} \arrow[r, "\emptyset"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\Longrightarrow"] & !\mathbb{A} \arrow[d, "\oplus^{r}"] \\
\mathbf{I} \otimes \mathbf{I} \arrow[r, "\emptyset \otimes \emptyset"'] & !\mathbb{A} \otimes !\mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
!\mathbb{A} \otimes !\mathbb{A} \arrow[r, "\emptyset^{r} \otimes \emptyset^{r}"] \arrow[d, "\oplus"'] \arrow[dr, phantom, "\Longleftarrow"] & \mathbf{I} \otimes \mathbf{I} \arrow[d, "\cong"] \\
!\mathbb{A} \arrow[r, "\emptyset^{r}"'] & \mathbf{I}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & !\mathbb{A} \arrow[dr, "\emptyset^{r}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathbf{I} \arrow[ur, "\emptyset"] \arrow[rr, "\mathrm{I}"'] & {} & \mathbf{I}
\end{tikzcd}
$$

are invertible.

Monoidal laws:

The canonical 2-cells

$$
\begin{tikzcd}
!\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{B} \arrow[r, "\oplus \otimes \mathrm{I}"] \arrow[d, "\mathrm{I} \otimes \mathrm{I} \otimes \oplus^{r}"'] & !\mathbb{A} \otimes !\mathbb{B} \arrow[r, "\mathrm{m}"] & !(\mathbb{A} \otimes \mathbb{B}) \\
!\mathbb{A} \otimes !\mathbb{A} \otimes !\mathbb{B} \otimes !\mathbb{B} \arrow[d, "\mathrm{I} \otimes \sigma \otimes \mathrm{I} \cong"'] & \Uparrow & \\
!\mathbb{A} \otimes !\mathbb{B} \otimes !\mathbb{A} \otimes !\mathbb{B} \arrow[rr, "\mathrm{m} \otimes \mathrm{m}"'] & & !(\mathbb{A} \otimes \mathbb{B}) \otimes !(\mathbb{A} \otimes \mathbb{B}) \arrow[uu, "\oplus"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{A} \otimes !\mathbb{B} \arrow[r, "\eta \otimes \mathrm{I}"] \arrow[d, "\mathrm{I} \otimes \eta^{r}"'] \arrow[dr, phantom, "\Longrightarrow"] & !\mathbb{A} \otimes !\mathbb{B} \arrow[d, "\mathrm{m}"] \\
\mathbb{A} \otimes \mathbb{B} \arrow[r, "\eta"'] & !(\mathbb{A} \otimes \mathbb{B})
\end{tikzcd}
$$

are invertible.

Comonad multiplication law:

The canonical 2-cell

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "\eta"] \arrow[d, "\cong"'] & !\mathbb{A} \arrow[r, "\mu^{r}"] \arrow[d, phantom, "\Uparrow"] & !!\mathbb{A} \\
\mathbb{A} \otimes \mathbf{I} \arrow[r, "\eta \otimes \emptyset"'] & !\mathbb{A} \otimes !\mathbb{A} \arrow[r, "\eta \otimes \mu^{r}"'] & !!\mathbb{A} \otimes !!\mathbb{A} \arrow[u, "\oplus"']
\end{tikzcd}
$$

is invertible.

Additive laws:

The canonical 2-cells

$$0 \Longrightarrow \emptyset^{\mathrm{r}} \cdot \eta : \ \mathbb{A} \longrightarrow \mathbf{I}$$

$$0 \Longrightarrow \eta^{\mathrm{r}} \cdot \emptyset : \ \mathbf{I} \longrightarrow \mathbb{A}$$

$$(\eta \otimes \emptyset) + (\emptyset \otimes \eta) \Longrightarrow \oplus^{\mathrm{r}} \cdot \eta : \ \mathbb{A} \longrightarrow !\mathbb{A} \otimes !\mathbb{A}$$

$$(\eta^{\mathrm{r}} \otimes \emptyset^{\mathrm{r}}) + (\emptyset^{\mathrm{r}} \otimes \eta^{\mathrm{r}}) \Longrightarrow \eta^{\mathrm{r}} \cdot \oplus : \ !\mathbb{A} \otimes !\mathbb{A} \longrightarrow \mathbb{A}$$

are invertible.

THE LANDSCAPE

substitution tensor product

Cartesian Closure

[FGHW] preprint

Differential & Quantum structure [FOSSACS05, CKC06]

$CoK\ell(!) \simeq Esp$

$\vdash$

$(\cdot)^{\#}$

Af --- and domain theory?

linear ! comonad

$Mat \simeq Lin$

compact closed bicategory [DS]

$\cong$

! monad lifting

$\underline{K\ell}(\hat{\cdot})$

$\dashv$

bicategorical lim/colim coincidence [CF]

dist. law

$\underline{Maps}(Mat) \simeq\ !\,Cat \xrightarrow{\widehat{(\cdot)}} CAT$

Kleisli structure [FGHW]

$\dashv$

CommMon(Cat)

$\cong$

SymmMonCat

$$Mat(\mathbb{K},\mathbb{L}) = [\mathbb{K},\hat{\mathbb{L}}] \cong CoCont[\hat{\mathbb{K}},\hat{\mathbb{L}}] \cong Lin(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

Maps = left adjoints ... ($\Downarrow$ $RR^{o} \leq id$ & $R^{o}R \geq id$ / R is functional)

ANALYTIC FUNCTORS
~ DOMAIN-THEORETIC ASPECT ~

Thm: [Joyal] A functor Set → Set is analytic iff it preserves direct and inverse limits, and quasi-pullbacks

$$\begin{array}{ccc} Q & \longrightarrow & Y \\ \downarrow & \text{qpb} & \downarrow \\ X & \longrightarrow & Z \end{array}$$

iff$_{\text{def}}$ $Q \to X \times_Z Y$ is regular epi

NB: In a partial order, qpb = pb = bounded infs

⇝ STABLE domain theory [Berry]

THE NOTION OF APPROXIMATION

$$\mathrm{Esp}(\mathbb{K},\mathbb{L}) \overset{?}{\simeq} \mathcal{A}f(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

$$\begin{tikzcd} !\mathbb{K} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{L} \end{tikzcd} \leftrightarrow \begin{tikzcd} \hat{\mathbb{K}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow ?"] & \hat{\mathbb{L}} \end{tikzcd}$$

Thm:

$$\mathrm{Esp}(\mathbb{1},\mathbb{1}) \simeq \mathrm{Set}^{\mathrm{Bij}} \overset{[\text{Joyal}]}{\simeq} \mathcal{A}f_{qc}(\mathrm{Set},\mathrm{Set})$$

quasi-cartesian natural transformations (cf. stable order)

However:

$$\begin{tikzcd} !\Sigma \arrow[r, bend left, "(\vec{\top})"] \arrow[r, bend right, "(\vec{\bot})"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{1} \end{tikzcd} \mapsto \begin{tikzcd} \hat{\Sigma} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \text{NOT } qc"] & \mathrm{Set} \end{tikzcd}$$

ANALYTIC FUNCTORS & DOMAIN THEORY

Thm: For $\mathbb{G}$ and $\mathbb{H}$ groupoids:

$F : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ is analytic

iff

F preserves direct and inverse limits and quasi-pullbacks

iff

F preserves direct limits and wide quasi pullbacks

Thm: $\mathrm{Esp}(\mathbb{G}, \mathbb{H}) \simeq \mathcal{A}f_{pqc}(\widehat{\mathbb{G}}, \widehat{\mathbb{H}})$

pointwise quasi cartesian

THE LANDSCAPE

substitution : tensor product

Cartesian Closure

[FGHW] preprint

Differential & Quantum structure [FOSSACS 05, CKC06]

Enumerative combinatorics

Domain theory
- fixed points
- graph models

$$
\begin{tikzcd}
\mathrm{CoK\ell}(!) \arrow[r, phantom, "\simeq"] \arrow[dr, shift left] \arrow[dr, phantom, "\dashv"] & \mathrm{Esp} \arrow[rr, leftrightarrow] & & \mathrm{Esp}_g \arrow[r, phantom, "\simeq"] & \mathcal{A}f_g \\
& \mathrm{Mat} \arrow[ul, shift left] \arrow[u, "(-)^\#"'] \arrow[loop left, "!"] \arrow[r, phantom, "\simeq"] & \mathrm{Lin} \arrow[dr, phantom, "\cong"] & \mathrm{Mat}_g \arrow[u] \arrow[ll, leftrightarrow] & \\
& & & \mathrm{K\ell}(\hat{\ }) \arrow[loop right, "!"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\
& \mathrm{Maps}(\mathrm{Mat}) \arrow[uu] \arrow[rr, phantom, "\simeq"] & & !G\,\mathrm{Cat} \arrow[u, shift left] \arrow[r, "\hat{(\cdot)}"] \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & \mathrm{CAT} \\
& & & \mathrm{CommMon}(\mathrm{Cat}) \arrow[u, shift right] \arrow[d, phantom, "\cong"] & \\
& & & \mathrm{SymmMonCat} &
\end{tikzcd}
$$

linear ! comonad

compact closed bicategory [DS]

bicategorical lim/colim coincidence [CF]

strong compact closure [CKC06]

! monad lifting

dist. law

Kleisli structure [FGHW]

$$\mathrm{Mat}(\mathbb{K},\mathbb{L}) = [\mathbb{K},\hat{\mathbb{L}}] \cong \mathrm{CoCont}[\hat{\mathbb{K}},\hat{\mathbb{L}}] \cong \mathrm{Lin}(\hat{\mathbb{K}},\hat{\mathbb{L}})$$

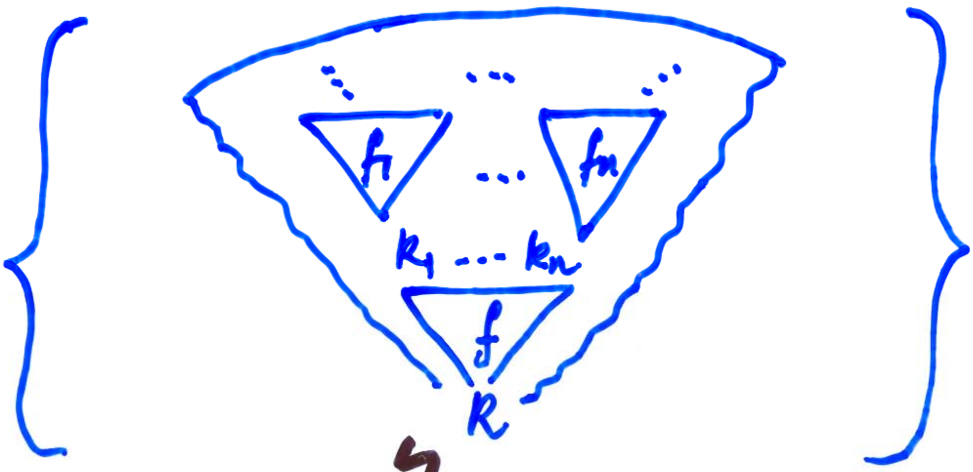
Maps = left adjoints ... $\left(\Leftrightarrow \begin{array}{l} RR^\circ \subseteq \mathrm{id} \;\&\; R^\circ R \supseteq \mathrm{id} \\ R \text{ is functional} \end{array} \right)$

strong compact closure [AC] $\rightsquigarrow$ adjoints as in linear algebra

FIXED POINTS

$$\frac{F : \mathbb{K} \mathrel{!\!\rightarrow} \mathbb{K} \quad \text{in Esp}}{fix(F) \in \hat{\mathbb{K}}}$$

$fix(F)_k =$



equivalence classes of F-labelled k-rooted trees

A GRAPH MODEL OF THE DIFFERENTIAL λ-CALCULUS

[EHRHARD & REGNIER]

$$!U \times U \cong U \quad \text{in } \mathrm{Gpd}$$

gives

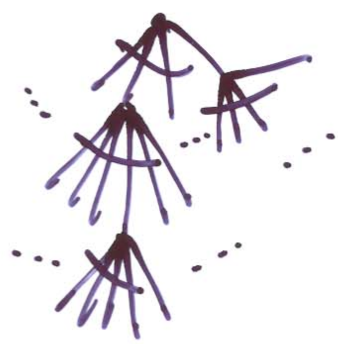
$$[U \mathrel{\hat{\to}} U] \cong \hat{U} \quad \text{in } \mathit{df}_g$$

Eg:

$$\begin{tikzcd}
1 & !1 \times 1 \cong \mathrm{Bij} \arrow[l] & !\mathrm{Bij} \times \mathrm{Bij} \arrow[l] & \cdots \arrow[l] \\
 & U \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

final coalgebra

groupoid of trees



with isomorphisms