Pseudo-Distributive Laws and a Unified Framework for Variable Binding

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Abstract

This thesis provides an in-depth study of the properties of pseudo-distributive laws motivated by the search for a unified framework to model substitution and variable binding for various different types of contexts; in particular, the construction presented in this thesis for modelling substitution unifies that for cartesian contexts as in the work by Fiore *et al*. and that for linear contexts by Tanaka.

The main mathematical result of the thesis is the proof that, given a pseudo-monad S on a 2-category $\mathbb{C}$, the 2-category of pseudo-distributive laws of S over pseudo-endofunctors on $\mathbb{C}$ and that of liftings of pseudo-endofunctors on $\mathbb{C}$ to the 2-category of the pseudo-algebras of S are equivalent. The proof for the non-pseudo case, i.e., a version for ordinary categories and monads, is given in detail as a prelude to the proof of the pseudo-case, followed by some investigation into the relation between distributive laws and Kleisli categories. Our analysis of distributive laws is then extended to pseudo-distributivity over pseudo-endofunctors and over pseudo-natural transformations and modifications. The natural bimonoidal structures on the 2-category of pseudo-distributive laws and that of (pseudo)-liftings are also investigated as part of the proof of the equivalence.

Fiore *et al*. and Tanaka take the free cocartesian category on 1 and the free symmetric monoidal category on 1 respectively as a category of contexts and then consider its presheaf category to construct abstract models for binding and substitution. In this thesis a model for substitution that unifies these two and other variations is constructed by using the presheaf category on a small category with structure that models contexts. Such structures for contexts are given as pseudo-monads S on *Cat*, and presheaf categories are given as the free cocompletion (partial) pseudo-monad T on *Cat*, therefore our analysis of pseudo-distributive laws is applied to the combination of a pseudo-monad for contexts with the cocompletion pseudo-monad T. The existence of such pseudo-distributive laws leads to a natural monoidal structure that is used to model substitution. We prove that a pseudo-distributive law of S over T results in the composite TS again being a pseudo-monad, from which it follows that the category $TS1$ has a monoidal structure, which, in our examples, models substitution.

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Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Miki Tanaka*)

Table of Contents

Chapter 1

Introduction

1.1 History and motivations

1.1.1 Variable binding and substitution

Issues surrounding variable binding and substitution have always been an important research topic throughout the history of computer science. Variable binding is a situation where a variable becomes associated with another symbol, typically denoting an operation, or, conceptually equivalently, a *function*, and as the result of this association, the variable loses its full distinction as a symbol and becomes only distinguishable relative to the symbol with which it is associated. Drawing an example from some simple mathematics, consider an expression $x+a$, where both x and a denote variables, although the implicit intention in the choice of symbols is rather clear here. Then suppose we name this expression f using $=$ and at the same time associate the symbol x with this symbol f. A typical representation of this situation is the expression $f(x) = x+a$. We say that the variable x is *bound* in the expression $x+a$ on the right hand side. We can apply the same discussion to an expression $y+a$ to obtain the expression $f(y) = y+a$. Then these two resulting expressions are indistinguishable, in the sense that both x and y are associated with f in exactly the same way, and hence, having lost the distinction as symbols they render the two expressions indistinguishable. This phenomenon has traditionally called α-equivalence in the study of λ-calculi, where the function $f(x) = x+a$ is namelessly denoted by an expression $\lambda x.x+a$. Again, we say that the variable x is bound by λ and call x a bound variable.

We have yet to define the precise meaning of "associating" a symbol with another, which can be done in more than one way as we see later, but the most common way is to regard such an f as higher-order, with the associated symbols as formal parameters for the function.

Now, with a function and formal parameters, the next thing to consider is applying an argument to a function. Given a function $f(x) = x+a$ and an argument, say, b, the value $f(b)$ of this argument applied to this function is $b+a$, where the actual argument b is substituted for the formal parameter x. In the λ-calculus terminology, the application of an argument to a function is denoted by juxtaposition, i.e., in this case, $(\lambda x.x+a)b$. Substituting the argument b for the bound variable x is represented as $(x+a)[b/x]$, which is equal to the value of the application $b+a$. The representation $M[N/x]$ for expressions M and N and a variable x should read "the expression obtained as the result of substituting N for all the x's appearing in M". We defer the precise definition of substitution for later, but what one has to be cautious in the definition is to consistently take care of situations where variables appearing in the expression to be substituted might become bound as the result of substitution, for example, consider the case of $(\lambda y.x+y)[y/x]$. When the substitution is interpreted as application of an argument to a function, this should not be allowed in general, and $M[N/x]$ should be defined accordingly. This is usually done by using α-conversion, i.e., by renaming the relevant bound variables in the function body.

Manipulation of symbols at this level of complexity presents unexpectedly difficult problems particularly when we want to process such expressions automatically, i.e., using computers, because one needs to formulate precisely and properly how symbols are associated and how and when symbols are distinguished or not distinguished. Moreover, this needs to be done in a "good" way in order for us to make use of the syntactic nature of the expressions. Plenty of effort has been put into this area of research to establish a good model of variable binding and substitution [dB72, Sto88].

Recently there has been some new developments in the direction of category-theoretic models. In [FPT99, Hof99] presheaf categories were used as the basis for the representation of syntax with variable binding. Meanwhile, Pitts and Gabbay [GP99] proposed the use of Fraenkel-Mostowski set theory, and then the Schanuel topos. Our focus in the following is the first direction, which was also studied in a modified setting for linear binding by Tanaka [Tan00].

1.2 Developments so far

Around 1970, Kelly introduced the notion of club [Kel72a] in order to deal with coherence theorems for category theory.

We will not go into any details on clubs here, except to remark that almost thirty years later Fiore *et al*. [FPT99] used a structure that is a variant of clubs, to provide binding algebras to model variable binding and monoidal structure to model substitution. Using $\mathbb{F}$, the category freely generated from 1 by adding finite coproducts, as the category of contexts, they built their model of variable binding, called binding algebra, on the presheaf category $Set^{\mathbb{F}}$. The main analogy is that instead of algebras over sets as in universal algebra here one considers binding algebras over variable sets, which are modelled by presheaves. The presheaf category $Set^{\mathbb{F}}$ inherits finite product structure from $\mathbb{F}^{op}$. This structure is a restriction of Kelly's club and it is a conceptual improvement in choice, for the application to computer science.

The kind of binding discussed in that paper is the one which is most common, but it is natural to think of other variations in binders, as in [Tan00], where linear binders are considered. In that paper, binding algebra and substitution monoids are adapted to the case of linear binders, using the free symmetric monoidal category $\mathbb{P}$ on 1. The resulting structure is again closely related to Kelly's original clubs, being a variant of his clubs over $\mathbb{P}$.

Having seen these developments in modelling different kinds of binders, Power [Pow03] recently described an idea of unifying these structures for different kinds of binders by providing a category-theoretic framework along the lines of [Tan00]. That not only includes the two examples, but it also allows one to consider a wider variety of examples, including, in particular, that given by that Logic of Bunched Implications [Pym02].

1.3 The aim of this thesis

The paper [Pow03] is based on the definition of a pseudo-distributive law between pseudo-monads given in [Mar99]. However, the definition given in [Mar99] is incomplete, in the sense that one of the coherence axioms is missing and the duality in those axioms is not reflected in the presentation.

The aim of this thesis is to provide a solid technical foundation for the above idea by Power by studying in detail pseudo-distributive laws between pseudo-monads and giving their full coherence axioms. A complete and definitive definition of pseudo-distributive laws is given, together with a detailed investigation of some of their properties, followed by a brief investigation of substitution as a main example of its use, in particular in association with cartesian binders, linear binders and binders of Bunched Implications.

For ordinary monads, given two monads S and T on a category $\mathbb{C}$, a distributive law δ of S over T is a natural transformation $\delta : ST \to TS$ such that certain commutative diagrams involving the multiplications and units of both S and T are satisfied. But what we need is the notion of pseudo-distributive law rather than that of distributive law. The "pseudo-ness" arises as follows: Take the 2-monad T_{fp} on *Cat* for finite product structure which will be discussed in Section 8.1. Given a small category $\mathcal{C}$, $T_{fp}\mathcal{C}$ is a free category with finite products on $\mathcal{C}$. Let FP be the category of small categories with finite products and product-preserving functors. We claim that FP is equivalent, not to the category of T_{fp}-algebras, but to the category of pseudo-T_{fp}-algebras. There is an obvious forgetful functor U from FP to *Cat*. Now consider if this U has a left adjoint. If there exists a left adjoint F, since F preserves colimits and *Cat* has an initial object 0, the value $F0$ should be an initial object in FP. But this is not the case because FP does not have an initial object. For consider the category *Iso* of a pair of objects and an isomorphism between them. Any category with finite products has at least two finite product preserving functors into it. Therefore it is essential here to have pseudo-ness in the structure, more precisely, the notion of pseudo-maps is crucial here. We choose to deal with pseudo-algebras, too.

The central result about ordinary distributive laws is the equivalence between a distributive law $\delta : ST \to TS$ and a lifting of T to S-*Alg*. But in our examples, what we have is *Ps-S-Alg*, the 2-category of pseudo-S-algebras. So, correspondingly, we must generalise from an ordinary distributive law to a pseudo-distributive law. For a pseudo-distributive law, we need to consider a pseudo-natural transformation together with invertible modifications replacing equality in the commutative diagrams, and these modifications are subject to several coherence conditions, which usually are very complex.

Now consider the composite TS determined by a pseudo-distributive law $ST \to TS$. Although the examples of pseudo-monads that we study later in this thesis are actually 2-monads regarded as pseudo-monads, we cannot avoid pseudo-monads because the composite of 2-monads has the structure of a pseudo-monad, not of a 2-monad. This result is essential in our construction, hence we choose to develop our discussion at the level of pseudo-monads from the start.

We study the properties of pseudo-distributive laws by starting from the non-pseudo version of them; we first give proofs of the properties of ordinary distributive laws, and then we extend the discussions to the case of pseudo-distributive laws. One cannot fail to notice that the commutative diagrams appearing in the proofs for the non-pseudo case become part of the construction in the pseudo-case, i.e., are replaced by pieces of data such as 2-cells and modifications, and that what needs to be proved then is coherence for those data.

In [Mar99] Marmolejo gave a definition of a pseudo-distributive law between pseudo-monads. However this was done in a very specific setting, namely, Gray-enriched categories, where Gray is the symmetric monoidal category whose underlying category is 2-*Cat* with tensor product [GPS95]. In the paper he gave nine coherence axioms, but most of these are described in a way for which the duality among these axioms is not easily understood. We have worked out a better and definitive definition of a pseudo-distributive law in a generic 2-categorical setting, as shown in Chapter 7 including a coherence axiom which was missing in Marmolejo's paper.

Having defined the pseudo-distributive law in full, it is necessary to have a detailed discussion of how the two pseudo-monads and their pseudo-algebras interact under the existence of a pseudo-distributive law. More specifically, the facts of interest here are that to give a pseudo-distributive law δ of S over T is equivalent to give a lifting of

T to *Ps-S-Alg*, the 2-category of pseudo-S-algebras, or to give an extension of S to $Kl(T)$, the Kleisli bicategory of T, and that the functor TS acquires the structure of a pseudo-monad. We have provided a precise description and proofs of those properties for the case of ordinary monads with a distributive law, which is reformulated into the pseudo setting for the pseuod-algebra case by carefully replacing the commuting diagrams with invertible modifications. Our proof also shows that in the non-pseudo setting the equivalence is in fact an isomorphism.

To provide the unifying framework for substitution, it is also necessary to introduce the notion of pseudo-strengths of a pseudo-monad and study their properties. This is one of the main results given in Chapter 8. We present the definition of a pseudo-strength with ten coherence axioms; one can find many similarities between these axioms and those of a pseudo-distributive law, which reflects the fact that a pseudo-strength can be regarded as a special case of a pseudo-distributive law.

We present the unifying framework for substitution as one example of applications of our analysis on pseudo-distributivity. The construction is based on the existence of a pseudo-distributive law of a pseudo-monad S over a pseudo-monad T, where S is one of the pseudo-monads that gives a category which models a certain type of context, while T is the (partial) pseudo-monad for free cocompletion. Here we need to address the size issue of this particular pseudo-monad on *Cat* because the free cocompletion of a small category $\mathcal{C}$ is not small in general. More detailed discussion is found in Section 8.1.

There are other areas where the analysis of pseudo-distributive laws in this thesis can be applied. One of them is the study of concurrency and bisimulation by Winskel and Cattani [WC04] using open maps and profunctors; the structure used there involves pseudo-comonads and Kleisli constructions. The analysis of pseudo-distributive laws in this thesis can be easily applied to the case of pseudo-comonads.

1.4 Outline

Chapter 2 provides the basic knowledge required for the rest of the thesis. Section 2.1 contains a quick summary of several topics from ordinary category theory, including monads and their algebras, adjunctions, monoidal categories and monoids. Then the notion of 2-categories and related notions such as cells, 2-functors, 2-natural transformations are defined in Section 2.2, followed by the definition of pseudo-functors, pseudo-natural transformations, modifications, and then finally pseudo-monads and their morphisms in Section 2.3. A brief introduction to the notion of pasting is also included. Section 2.4 introduces the notions of pseudo-algebras of a pseudo-monad, pseudo-maps between pseudo-algebras, and 2-cells between pseudo-maps, all of which together define the 2-category of pseudo-algebras. The last section contains the definitions of bicategories and bimonoidal bicategories.

Chapter 3 is devoted to the study of the properties of distributive laws in ordinary categories, which will be extended to the pseudo case in 2-categories in later chapters. It starts with the definition of distributivity of a monad S over an endofunctor H, and also over a natural transformation in Section 3.1. Then we introduce the notion of a lifting of an endofunctor H to the category of S-algebras in Section 3.2. In the following three sections it is proved that the category $\mathsf{Dist}^{\mathsf{S}}$ of distributive laws of a monad is isomorphic to $\mathsf{Lift}_{S\text{-}Alg}$, the category of liftings of endofunctors to the category of algebras of the monad.

In order to prove the similar isomorphism for distributive laws over a monad rather than an endofunctor, we need the notion of lifting of a monad T to a monad $\widehat{T}$ on S-Alg: the multiplication $\widehat{\mu} : \widehat{T}^2 \to \widehat{T}$ of $\widehat{T}$ should be given by the lifting of μ as a natural transformation. Consequently, the proof of the isomorphisms requires some analysis of the relation between $\widehat{T}^2$ and $\widehat{T^2}$ and also how that relates to distributive laws. We investigate this issue in Section 3.6 for the case of H^2, where H is an endofunctor. We establish the relationship between the square of a lifting of H and a particular distributive law of a monad over H^2. This leads to the discussion in Section 3.7 on distributive laws of a monad over a monad. The last section (Section 3.8) in Chapter 3 studies the properties of the composite TS under the existence of a distributive law of a monad S over a monad T. We see that in this case the functor TS is a monad.

Chapter 4 is in a sense dual to Chapter 3; the relationship between distributive laws over a monad T and the Kleisli category $Kl(T)$ of the monad T is established. First the definitions of the notion of distributive laws of endofunctors over a monad T are given in Section 4.1 and an extension of an endofunctor to $Kl(T)$ is defined in Section 4.2. Then in Sections 4.3, 4.4 and 4.5 the proof that there is an isomorphism between the category of distributive laws of endofunctors over a monad and that of extensions of endofunctors to the Kleisli category of the monad is given. In the following section (Section 4.6) we develop an analysis similar to that in Section 3.6 of the relationship between an extension of H^2 and distributive laws. The rest of the chapter contains the proof that the category of distributive laws of monads over a monad T is also isomorphic to the category of extensions of monads to the Kleisli category $Kl(T)$ of T. We conclude the chapter by stating a theorem that summarises the results in Chapter 3 and 4.

Chapter 5 The discussion in the first five sections in Chapter 3 is extended to the pseudo-setting, by systematically replacing the commuting diagrams with invertible modifications or 2-cells. In Section 5.1 the definition of pseudo-distributivity of a pseudo-monad S over pseudo-endofunctors, pseudo-natural transformations, and modifications are given, and it is shown that these data constitute a 2-category called $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. Similarly, in Section 5.2, the liftings of pseudo-endofunctors, pseudo-natural transformations and modifications to the 2-category of pseudo-S-algebras are defined, and they define a 2-category $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$. One can define pseudo-functors between these two 2-categories, as shown in the following two sections (Section 5.4, 5.3), which define an equivalence of 2-categories (Section 5.5).

Chapter 6 is the pseudo-version of Section 3.6 (and also of Section 4.6), expanded and generalised. The motivation for this chapter is the same as that for those sections. The properties of H^2 investigated for ordinary endofunctors are in fact derived from the monoidal structures on $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{S\text{-}Alg}$, and the isomorphism between them preserves those structures (Section 6.1). In the pseudo-case, in Section 6.2, the situation is much more complex; the structure on $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ is a special case of bimonoidal structure. Still, the pseudo-functors that define an equivalence between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ preserve these structures, i.e., they are 2-strong bimonoidal 2-functors, to be precise (Section 6.3).

Chapter 7 is the pseudo-version of Section 3.7. The precise definition of a pseudo-distributive law of a pseudo-monad over a pseudo-monad is given in Section 7.1, together with its complete set of coherence axioms. These define the 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$ of pseudo-distributive laws of a pseudo-monad S over pseudo-monads, which is a variant of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. Then, in Section 7.2, the 2-category $\mathsf{Lift}^{\mathsf{ps\text{-}monads}}_{Ps\text{-}S\text{-}Alg}$ of liftings of pseudo-monads to the 2-category of pseudo-S-algebras is defined. The equivalence of these two 2-categories is proved in Section 7.3. The existence of a pseudo-distributive law of a pseudo-monad S over a pseudo-monad T implies that the composite TS is again a pseudo-monad, and this together with a few more properties are stated and proved in Section 7.4.

Chapter 8 contains the main application of the theoretical development of the thesis, i.e., the construction of the generic substitution monoidal structure is given in depth. We start the chapter by examining several examples of pseudo-monads, including T_{fp} and T_{sm}, and their pseudo-algebras in Section 8.1 and examples of pseudo-distributive laws between them in Section 8.2. We also introduce the (partial) pseudo-monad T_{coc} for the free cocompletion and address the relevant size issues, too. After defining and studying the notion of strength for ordinary monads in Section 8.3, and that of pseudo-strength for pseudo-monads in Section 8.4, we show that an arbitrary pseudo-monad T on Cat yields a canonical monoidal structure on the category $T1$ in Section 8.5. The significance of that monoidal structure, as we explain as examples in Section 8.6, is that when T is the pseudo-monad $T_{coc}T_{fp}$, it yields precisely Fiore *et al.*'s substitution monoidal stucture, and likewise for Tanaka when T is $T_{coc}T_{sm}$. Moreover, at the level of generality proposed here, we can follow the main line of development of both pieces of work.

Chapter 9 summarises the thesis and discusses possible directions for future work.

Chapter 2

Preliminaries

This chapter contains definitions of category theoretic terms used in this thesis. These will serve to fix notation and also to remind readers of some basics, including monads and their algebras, 2-categories and 2-functors, and most importantly, pseudo-functors, pseudo-monads and pseudo algebras.

2.1 Monads and their algebras

In this section, the notions of a monad (ordinary) and its algebras are defined. After the definition of the category of algebras, *T-Alg*, we state several important results on the relationship between monads and adjunctions, which are used in Section 3.8. We finish the section with the definitions of monoidal categories and monoids in them; these are needed in Chapter 8.

Definition 2.1 (monad). A *monad* (T,μ,η) on a category $\mathcal{C}$ consists of a functor $T:\mathcal{C}\to\mathcal{C}$ and two natural transformations, the multiplication $\mu: T^2 \to T$ and the unit $\eta: Id_{\mathcal{C}} \to T$, such that the following diagrams, one for the associativity of μ and another for the left and right unity of η, commute:

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "id_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "id_T"] \\
& T &
\end{tikzcd}
$$

Definition 2.2 (monad morphism). Given monads (T,μ,η) and (T',μ',η'), a natural transformation $\alpha : T \to T'$ is called *a monad morphism from* (T,μ,η) *to* (T',μ',η') if the following diagrams commute:

$$\begin{tikzcd} T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu"'] & TT' \arrow[r, "\alpha T'"] & T'^2 \arrow[d, "\mu'"] \\ T \arrow[rr, "\alpha"] & & T' \end{tikzcd} \qquad \begin{tikzcd} Id \arrow[d, "\eta"'] \arrow[dr, "\eta'"] & \\ T \arrow[r, "\alpha"] & T' \end{tikzcd}$$

Definition 2.3 (algebras for a monad). Given a monad (T,μ,η) on $\mathcal{C}$, a *T-algebra* $\langle A,a\rangle$ is a pair consisting of an object A of $\mathcal{C}$ and an arrow $a : TA \to A$ of $\mathcal{C}$, called the *structure map* of the algebra, such that the following two diagrams, one called the associative law and the other the unit law, commute:

$$\begin{tikzcd} T^2A \arrow[r, "Ta"'] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"'] \arrow[dr, "id_T"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

A map $f : \langle A,a\rangle \to \langle A',a'\rangle$ of T-algebras is an arrow $f : A \to A'$ in $\mathcal{C}$ which makes the following diagram commute:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TA' \arrow[d, "a'"] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$

These data constitute the category *T-Alg* of T-algebras and T-algebra maps. There is the obvious forgetful functor G^T : *T-Alg* $\to \mathcal{C}$.

Now we state several important results about monads and adjunctions. Given a category $\mathcal{C}$ and a monad on it, there exists a canonical adjunction induced by this monad. On the other hand, an adjunction defined on $\mathcal{C}$ also defines a monad on $\mathcal{C}$.

Lemma 2.4 (a monad induced adjunction). *If (T,μ,η) is a monad on $\mathcal{C}$, then there exists an adjunction*

$$(F^T, G^T, \eta^T, \varepsilon^T) : \mathcal{C} \underset{G^T}{\overset{F^T}{\rightleftarrows}} T\text{-}Alg.$$

F^T sends an object A in $\mathcal{C}$ to the free T-algebra $\langle TA, \mu_A : T^2A \to TA\rangle$, η^T is the unit η of the monad, and the component of ε^T at a T-algebra $\langle A,h\rangle$ is h.

Lemma 2.5 (an adjunction defines a monad). *Any adjunction*

$$(F, G, \eta, \varepsilon) : \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

gives rise to a monad $(GF, G\varepsilon F, \eta)$ *on* $\mathcal{C}$.

The following lemma states that the composition of two adjunctions again defines an adjunction.

Lemma 2.6 (composition of adjoints). *Given two adjunctions*

$$(F, G, \eta, \varepsilon) : \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad (F', G', \eta', \varepsilon') : \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

the composite functors yield an adjunction

$$(F'F, GG', G\eta'F \cdot \eta, \varepsilon' \cdot F'\varepsilon G') : \mathcal{C} \underset{GG'}{\overset{F'F}{\rightleftarrows}} \mathcal{E}$$

Now we consider the relationship between an adjunction and the adjunction canonically induced by the monad that the adjunction defines.

Lemma 2.7 (comparison of adjunctions with algebras [Mac98]). *Let* $(F, G, \eta, \varepsilon)$ *be an adjunction, where* $F : \mathcal{C} \to \mathcal{D}$, *and* $T = (GF, G\varepsilon F, \eta)$ *the monad it defines in* $\mathcal{C}$. *Then there exists a unique functor* $K : \mathcal{D} \to T\text{-}Alg$ *such that* $G^T K = G$ *and* $KF = F^T$.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K"] & & T\text{-}Alg \\
& \mathcal{C} \arrow[ul, "F"] \arrow[ur, "F^T"'] &
\end{tikzcd}$$

The comparison functor K *is constructed as follows: for an object* A *and an arrow* $f : A \to B$ *in* $\mathcal{D}$,

$$KA = \langle GA, G\varepsilon_A \rangle \qquad Kf = Gf : \langle GA, G\varepsilon_A \rangle \to \langle GB, G\varepsilon_B \rangle.$$

In the rest of the section, we define the notion of a monoidal category with symmetry and closeness, and that of a monoid in a monoidal category.

Definition 2.8 (monoidal category). A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathcal{C}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object I of $\mathcal{C}$, and three natural isomorphisms α, λ and ρ, whose components are given as, for any objects A, B and C,

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
$$\lambda_A : I \otimes A \cong A$$
$$\rho_A : A \otimes I \cong A$$

such that the following two diagrams commute: for any A, B, C and D in $\mathcal{C}$,

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "id \otimes \alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha \otimes id"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes C) \arrow[rr, "\alpha"'] \arrow[dr, "id \oplus \lambda"'] & & (A \otimes I) \otimes C \arrow[dl, "\rho \oplus id"] \\
& A \otimes C &
\end{tikzcd}$$

Sometimes a third axiom $\lambda_I = \rho_I : I \otimes I \to I$ is included in the definition but this has been found redundant by Kelly [Kel64].

There exists a notion of morphisms between monoidal categories: a *strong monoidal functor* is a functor between monoidal categories with additional structure that preserves monoidal structure up to isomorphisms. For a precise definition see [Mac98].

Definition 2.9 (symmetry). A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is called *symmetric* if it is equipped with a natural isomorphism γ, whose components are given as, for any objects A, B in $\mathcal{C}$,

$$\gamma_{A,B} : A \otimes B \cong B \otimes A,$$

for which the following diagrams commute:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, "id"'] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "\gamma_{A,I}"] \arrow[dr, "\rho_A"'] & I \otimes A \arrow[d, "\lambda_A"] \\
& A
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "\alpha"] \arrow[d, "id\otimes\gamma"'] & (A\otimes B)\otimes C \arrow[r, "\gamma"] & C\otimes(A\otimes B) \arrow[d, "\alpha"] \\
A\otimes(C\otimes B) \arrow[r, "\alpha"'] & (A\otimes C)\otimes B \arrow[r, "\gamma\otimes id"'] & (C\otimes A)\otimes B
\end{tikzcd}$$

Definition 2.10 (closedness). A symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ is *closed* if, for any object A in $\mathcal{C}$, the functor $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a specified right adjoint $(\ )^A : \mathcal{C} \to \mathcal{C}$.

Definition 2.11 (monoid). A *monoid* (X, μ, η) in a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of an object X of $\mathcal{C}$, together with arrows $\mu : X \otimes X \to X$ and $\eta : I \to X$ such that the diagrams

$$\begin{tikzcd}
X\otimes(X\otimes X) \arrow[r, "\alpha"] \arrow[d, "id_X\otimes\mu"'] & (X\otimes X)\otimes X \arrow[r, "\mu\otimes id_X"] & X\otimes X \arrow[d, "\mu"] \\
X\otimes X \arrow[rr, "\mu"'] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes X \arrow[r, "\eta\otimes X"] \arrow[dr, "\lambda_X"'] & X\otimes X \arrow[d, "\mu"] & X\otimes I \arrow[l, "X\otimes\eta"'] \arrow[dl, "\rho_X"] \\
& X &
\end{tikzcd}$$

commute.

2.2 2-Categories

In addition to the objects and arrows that constitute an ordinary category, in a 2-category, extra structure is introduced which is defined beween arrows. We call such extra structures *2-cells*. Accordingly, objects and arrows are often called 0-cells and 1-cells, respectively. The notion of vertical and horizontal compositions play an important rôle in the definition (See [Mac98]). Constructions in 2-categories are often expressed using diagrams of a certain kind: typically, their vertices denote the 0-cells, arrows the 1-cells, and the areas delimited by arrows in a particular way denote 2-cells. Such diagrams are used extensively throughout in the rest of this thesis. For a detailed discussion of 2-categorical diagrams and the notion of *pasting*, first introduced by Bénabou in [Bén67], we refer to the papers [Pow90, KS74].

Definition 2.12 (2-category). A *2-category* $\mathbb{C}$ consists of the following data:

- a set $\mathbb{C}_0$ of objects, called *0-cells*.

- for each pair of 0-cells A and B, a category $\mathbb{C}(A,B)$ (hom-category), whose objects are called *1-cells* of $\mathbb{C}$ and whose arrows are called *2-cells* of $\mathbb{C}$.

- for each triple of 0-cells A, B and C, a functor

$$comp_{A,B,C} : \mathbb{C}(B,C) \times \mathbb{C}(A,B) \to \mathbb{C}(A,C)$$

  called *composition*.

- for each 0-cell A of $\mathbb{C}$, a functor

$$unit_A : \mathfrak{I} \to \mathbb{C}(A,A)$$

The functors *comp* and *unit* are subject to the commutativity of the following diagrams.

$$\begin{tikzcd}
\mathbb{C}(C,D)\times\mathbb{C}(B,C)\times\mathbb{C}(A,B) \arrow[r, "comp_{B,C,D}\times id_{\mathbb{C}(A,B)}"] \arrow[d, "id_{\mathbb{C}(C,D)}\times comp_{A,B,C}"] & \mathbb{C}(B,D)\times\mathbb{C}(A,B) \arrow[d, "comp_{A,B,D}"] \\
\mathbb{C}(C,D)\times\mathbb{C}(A,C) \arrow[r, "comp_{A,C,D}"] & \mathbb{C}(A,D)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}(A,B) \arrow[r, "id_{\mathbb{C}(A,B)}\times unit_A"] \arrow[d, "unit_B\times id_{\mathbb{C}(A,B)}"'] \arrow[dr, "id_{\mathbb{C}(A,B)}"] & \mathbb{C}(A,B)\times\mathbb{C}(A,A) \arrow[d, "comp_{A,A,B}"] \\
\mathbb{C}(B,B)\times\mathbb{C}(A,B) \arrow[r, "comp_{A,B,B}"'] & \mathbb{C}(A,B)
\end{tikzcd}$$

Here $\mathfrak{I}$ is the trivial category with one object 0 and its identity arrow (the terminal object in *Cat*). We denote the value $unit(0)$ in $\mathbb{C}(A,A)$ by id_A.

The fact that 1-cells are defined as objects of a category and 2-cells as arrows implies the associativity and the unit law for the vertical composition of 2-cells, and the two diagrams imply the associativity and the unit law for both the horizontal composition of 2-cells and the composition of 1-cells.

Notation 2.13. We denote the horizontal composition of 2-cells by $\circ$, and the vertical composition by $\cdot$. Composition in general is denoted simply by juxtaposition or sometimes by $\circ$.

Definition 2.14 (2-functor). Let $\mathbb{C}, \mathbb{D}$ be 2-categories. A *2-functor* F from $\mathbb{C}$ to $\mathbb{D}$ consists of

- a function $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$
- for each pair A, B of 0-cells, a functor $F_{A,B} : \mathbb{C}(A,B) \to \mathbb{D}(F_0A, F_0B)$

subject to the commutativity of the following diagrams;

$$\begin{tikzcd}
\mathbb{C}(B,C) \times \mathbb{C}(A,B) \arrow[r, "F_{B,C} \times F_{A,B}"] \arrow[d, "comp_{A,B,C}"'] & \mathbb{D}(FB,FC) \times \mathbb{D}(FA,FB) \arrow[d, "comp_{FA,FB,FC}"] \\
\mathbb{C}(A,C) \arrow[r, "F_{A,C}"] & \mathbb{D}(FA,FC)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{I} \arrow[d, "unit_A"'] \arrow[dr, "unit_{FA}"] & \\
\mathbb{C}(A,A) \arrow[r, "F_{A,A}"'] & \mathbb{D}(FA,FA)
\end{tikzcd}$$

The operation of F on 1-cells and 2-cells is defined in terms of functors on hom-categories. This means that, if we use F_1 and F_2 to denote the object part and the arrow part of the functor,

1. for a 2-cell $\alpha : f \to f' : A \to B$, $F_2\alpha$ is of type $F_1 f \to F_1 f' : F_0A \to F_0B$,
2. given another 2-cell β of type $f' \to f''$, $F_2(\beta \cdot \alpha) = (F_2\beta) \cdot (F_2\alpha)$ holds;

and

3. for the identity $id_f : f \to f$ on any 1-cell f, $F_2(id_f) = id_{F_1 f}$ holds.

In the second item above, the dot $\cdot$ denotes the vertical composition of 2-cells both in $\mathbb{C}(A,B)$ and $\mathbb{D}(F_0A, F_0B)$.

Moreover, the two diagrams above demonstrate the functoriality of F over *comp*, that is,

1. $F(\gamma \circ \alpha) = (F\gamma) \circ (F\alpha)$,

over the horizontal composition of 2-cells, and

2. $comp(Fg, Ff) = F(comp(g, f))$ and $F(id_A) = id_{FA}$,

over the composition of 1-cells.

Definition 2.15 (2-natural transformation). Let F, G be 2-functors from $\mathbb{C}$ and $\mathbb{D}$. A *2-natural transformation* α from F to G consists of a collection of 1-cells indexed by 0-cells of $\mathbb{C}$, such that, for each component $\alpha_A : FA \to GA$ at a 0-cell A, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}(A,B) \arrow[r, "F"'] \arrow[d, "G"'] & \mathbb{D}(FA,FB) \arrow[d, "\alpha_B \circ -"] \\
\mathbb{D}(GA,GB) \arrow[r, "- \circ \alpha_A"'] & \mathbb{D}(FA,GB)
\end{tikzcd}$$

Example 2.16. The 2-category *Cat*. The 0-cells are given by all small categories, 1-cells given by all functors between them, and 2-cells given by all natural transformations.

2.3 Pseudo-monads

From now on, we use the pasting of diagrams extensively. The two basic situations for pasting is

$$\begin{tikzcd}
{} \arrow[rr, "f"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\Downarrow \alpha"] & {} \arrow[dr, "u"] \arrow[d, phantom, "\Downarrow \beta"] & \\
& {} \arrow[ur, "h"] \arrow[rr, "v"'] & {} & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
& {} \arrow[rr, "u"] \arrow[dr, "k"] \arrow[d, phantom, "\Downarrow \gamma"] & {} \arrow[d, phantom, "\Downarrow \delta"] & {} \\
{} \arrow[ur, "f"] \arrow[rr, "g"'] & {} & {} \arrow[ur, "v"'] &
\end{tikzcd}$$

The first of these represents the 2-cell $\beta g \cdot u\alpha : uf \to uhg \to gv$ and the second is the 2-cell $v\gamma \cdot \delta f : uf \to vkf \to ug$, where the dot $\cdot$ denotes the horizontal composition. Therefore we give meaning to such composites as

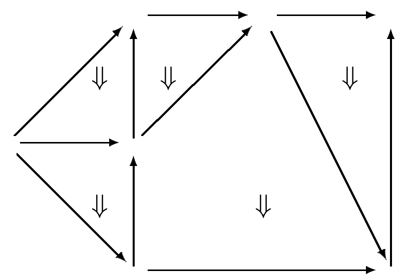
$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d] \\
{} \arrow[r] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
& {} \arrow[r] \arrow[d, phantom, "\Downarrow"] & {} \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
{} \arrow[ur] \arrow[r] & {} \arrow[ur] \arrow[rr] & {} & {}
\end{tikzcd}$$

If in a diagram such as

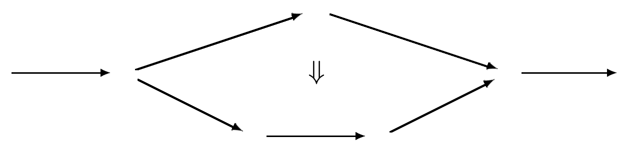
$$\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & {} \arrow[r, "g"] \arrow[d, "v"] & {} \arrow[d, "w"] \\
{} \arrow[r, "h"'] & {} \arrow[r, "k"'] & {}
\end{tikzcd}$$

one area has no 2-cells marked in it, it is to be understood that the identity 2-cell is meant, which implies that $wg = kv$.

One can generalise the pasting operation further, so as to give meaning to such multiple composite as



This is meant to indicate a vertical composite of horizontal composites of the form



There is usually a choice of the order in which the composites are taken, but the result is independent of this choice [KS74].

Now we give the definitions of pseudo-functor, pseudo-natural transformation, and modification.

Definition 2.17 (Pseudo-functor). Let $\mathbb{C}, \mathbb{D}$ be 2-categories. A *pseudo-functor* $(F, h, \overline{h})$ from $\mathbb{C}$ to $\mathbb{D}$ consists of the data for a 2-functor, plus

- for each triple A, B and C of 0-cells, an invertible natural transformation,

  $$h : comp_{FA,FB,FC} \circ (F \times F) \to F \circ comp_{A,B,C} : \mathbb{C}(B,C) \times \mathbb{C}(A,B) \to \mathbb{D}(FA,FC)$$

  whose component at (g, f) gives the isomorphism $Fg \circ Ff \cong F(g \circ f)$.

$$\begin{tikzcd}
\mathbb{C}(B,C) \times \mathbb{C}(A,B) \arrow[r, "F \times F"] \arrow[d, "comp_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow h"] & \mathbb{D}(FB,FC) \times \mathbb{D}(FA,FB) \arrow[d, "comp_{FA,FB,FC}"] \\
\mathbb{C}(A,C) \arrow[r, "F"'] & \mathbb{D}(FA,FC)
\end{tikzcd}$$

- for each 0-cell A, an invertible 2-cell $\overline{h} : unit_{FA} \to F(unit_A)$

$$\begin{tikzcd}
\mathcal{I} \arrow[d, "unit_A"'] \arrow[dr, "unit_{FA}"] \arrow[dr, phantom, shift right=3ex, "\Downarrow \overline{h}"] & \\
\mathbb{C}(A,A) \arrow[r, "F"'] & \mathbb{D}(FA,FA)
\end{tikzcd}$$

subject to the following three coherence axioms, expressed using the diagrams below:

(1)

$$\begin{tikzcd}
\mathbb{C}(C,D)\times\mathbb{C}(B,C)\times\mathbb{C}(A,B) \arrow[r, "F\times F\times F"] \arrow[d, "id_{\mathbb{C}(C,D)}\times comp_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow F\times h"] & \mathbb{D}(FC,FD)\times\mathbb{D}(FB,FC)\times\mathbb{D}(FA,FB) \arrow[d, "id_{\mathbb{D}(FC,FD)}\times comp_{FA,FB,FC}"] \\
\mathbb{C}(C,D)\times\mathbb{C}(A,C) \arrow[r, "F\times F"'] \arrow[d, "comp_{A,C,D}"'] \arrow[dr, phantom, "\Downarrow h"] & \mathbb{D}(FC,FD)\times\mathbb{D}(FA,FC) \arrow[d, "comp_{FA,FC,FD}"] \\
\mathbb{C}(A,D) \arrow[r, "F"'] & \mathbb{D}(FA,FD)
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathbb{C}(C,D)\times\mathbb{C}(B,C)\times\mathbb{C}(A,B) \arrow[r, "F\times F\times F"] \arrow[d, "comp_{B,C,D}\times id_{\mathbb{C}(A,B)}"'] \arrow[dr, phantom, "\Downarrow h\times F"] & \mathbb{D}(FC,FD)\times\mathbb{D}(FB,FC)\times\mathbb{D}(FA,FB) \arrow[d, "comp_{FB,FC,FD}\times id_{\mathbb{D}(FA,FB)}"] \\
\mathbb{C}(B,D)\times\mathbb{C}(A,B) \arrow[r, "F\times F"'] \arrow[d, "comp_{A,B,D}"'] \arrow[dr, phantom, "\Downarrow h"] & \mathbb{D}(FB,FD)\times\mathbb{D}(FA,FB) \arrow[d, "comp_{FA,FB,FD}"] \\
\mathbb{C}(A,D) \arrow[r, "F"] & \mathbb{D}(FA,FD)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
\mathbb{C}(A,B) \arrow[d, "unit_B\times id"'] \arrow[dr, "unit_{FB}\times F"] \arrow[dr, phantom, shift right=3ex, "\Downarrow \overline{h}\times id_F"] & \\
\mathbb{C}(B,B)\times\mathbb{C}(A,B) \arrow[r, "F\times F"'] \arrow[d, "comp_{A,B,B}"'] \arrow[dr, phantom, "\Downarrow h"] & \mathbb{D}(FB,FB)\times\mathbb{D}(FA,FB) \arrow[d, "comp_{FA,FB,FB}"] \\
\mathbb{C}(A,B) \arrow[r, "F"'] & \mathbb{D}(FA,FB)
\end{tikzcd} \quad = \quad id$$

(3)
$$
\begin{tikzcd}
\mathbb{C}(A,B) \arrow[d, "id \times unit_A"'] \arrow[drr, "F \times unit_{FA}"] & & \\
\mathbb{C}(A,B) \times \mathbb{C}(A,A) \arrow[r, "F \times F"'] \arrow[d, "comp_{A,A,B}"'] \arrow[ur, phantom, "\Downarrow id_F \times \overline{h}"] & \mathbb{D}(FA,FB) \times \mathbb{D}(FA,FA) \arrow[d, "comp_{FA,FA,FB}"] \arrow[dl, phantom, "\Downarrow h"] \\
\mathbb{C}(A,B) \arrow[r, "F"'] & \mathbb{D}(FA,FB)
\end{tikzcd}
\quad = \quad id
$$

Definition 2.18 (Pseudo-natural transformation). Let $F = (F, h, \overline{h})$ and $G = (G, k, \overline{k})$ be pseudo-functors from $\mathbb{C}$ to $\mathbb{D}$. A *pseudo-natural transformation* α from F to G consists of the following data:

- for each 0-cell A, a 1-cell $\alpha_A : FA \to GA$,
- for each pair A, B of 0-cells, an invertible natural transformation $\alpha^{A,B}$, called *pseudo-naturality* of α,

$$\alpha^{A,B} : (G(-) \circ \alpha_A) \to (\alpha_B \circ F(-)) : \mathbb{C}(A,B) \to \mathbb{D}(FA, GB),$$

  whose components are 2-cells in $\mathbb{D}(FA, GB)$, indexed by 1-cells in $\mathbb{C}(A,B)$.

and subject to the coherence conditions expressed in the diagrams below: for every composable pair of 1-cells $f : A \to B$ and $g : B \to C$,

$$
\begin{tikzcd}
 & & GB \arrow[drr, "Fg"] & & \\
GA \arrow[urr, "Gf"] & \Downarrow \alpha_f^{A,B} & & \Downarrow \alpha_g^{B,C} & GC \\
 & & FB \arrow[uu, "\alpha_B"] \arrow[drr, "Fg"] & & \\
FA \arrow[uu, "\alpha_A"] \arrow[urr, "Ff"] \arrow[rrrr, "F(gf)"'] \arrow[rrrr, phantom, bend left=15, "\Downarrow h_{(f,g)}"] & & & & FC \arrow[uu, "\alpha_C"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & GB \arrow[dr, "Fg"] \arrow[d, phantom, "\Downarrow k_{(f,g)}"] & \\
GA \arrow[ur, "Gf"] \arrow[rr, "G(gf)"] & {} & GC \\
FA \arrow[u, "\alpha_A"] \arrow[rr, "F(gf)"'] \arrow[urr, phantom, "\Downarrow \alpha_{gf}^{A,C}"] & & FC \arrow[u, "\alpha_C"']
\end{tikzcd}
$$

and the component of $\alpha_{A,B}$ at id_A

$$
\begin{tikzcd}
FA \arrow[r, "\alpha_A"'] \arrow[d, "F(id_A) \cong id_{FA}"'] \arrow[dr, phantom, "\Downarrow \alpha^{A,B}_{id_A}"] & GA \arrow[d, "G(id_A) \cong id_{GA}"] \\
FA \arrow[r, "\alpha_A"] & GA
\end{tikzcd}
$$

is equal to the 2-cell $id_{\alpha_A} : \alpha_A \to \alpha_A$ in $\mathbb{D}(FA, GA)$.

The pseudo-naturality $\alpha^{A,B}$ is expressed in the following diagram:

$$\begin{tikzcd}
\mathbb{C}(A,B) \arrow[r, "G"'] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \alpha^{A,B}"] & \mathbb{D}(GA,GB) \arrow[d, "- \circ \alpha_A"] \\
\mathbb{D}(FA,FB) \arrow[r, "\alpha_B \circ -"] & \mathbb{D}(FA,GB)
\end{tikzcd}$$

Given an arrow f in $\mathbb{C}(A,B)$, the component $\alpha_f^{A,B}$ at f is a 2-cell

$$\alpha_f^{A,B} : Gf \circ \alpha_A \to \alpha_B \circ Ff \qquad \begin{tikzcd}
FA \arrow[r, "\alpha_A"'] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \alpha_f^{A,B}"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha_B"] & GB
\end{tikzcd}$$

The naturality of $\alpha^{A,B}$ amounts to the following equality: given arrows $f, f' : A \to B$ and a 2-cell $\gamma : f \to f'$ in $\mathbb{C}$,

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"'] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \alpha_f^{A,B}"] & GA \arrow[d, "Gf"', bend right] \arrow[d, "Gf'", bend left] \arrow[d, phantom, "\overset{\Longrightarrow}{G\gamma}"] \\
FB \arrow[r, "\alpha_B"] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\alpha_A"'] \arrow[d, "Ff"', bend right] \arrow[d, "Ff", bend left] \arrow[d, phantom, "\overset{\Longrightarrow}{F\gamma}"] \arrow[dr, phantom, "\Downarrow \alpha_{f'}^{A,B}"] & GA \arrow[d, "Gf'"] \\
FB \arrow[r, "\alpha_B"] & GB
\end{tikzcd}$$

Notation 2.19. We usually suppress the superscripts ^{A,B} whenever they are clear from the context.

Definition 2.20 (Modification). Let α and β be pseudo-natural transformations. A *modification* χ from α to β consists of a collection of 2-cells $\{\chi_A : \alpha_A \to \beta_A\}$ indexed by 0-cells A of $\mathbb{C}$, such that, for every 1-cell $f : A \to B$, the following holds:

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \alpha_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, bend left, "\alpha_B"] \arrow[r, bend right, "\beta_B"'] \arrow[r, phantom, "\Downarrow \chi_B"] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, bend left, "\alpha_A"] \arrow[r, bend right, "\beta_A"'] \arrow[r, phantom, "\Downarrow \chi_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \beta_f"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\beta_B"'] & GB
\end{tikzcd}$$

Now we are ready to give the definition of pseudo-monad:

Definition 2.21 (Pseudo-monad). A *pseudo-monad* $T = (T, \mu, \eta, \tau, \lambda, \rho)$ on a 2-category $\mathbb{C}$ consists of

- a pseudo-functor $T : \mathbb{C} \to \mathbb{C}$
- a pseudo-natural transformation $\mu : T^2 \to T$
- a pseudo-natural transformation $\eta : Id_{\mathbb{C}} \to T$
- an invertible modification $\tau : \mu \circ T\mu \to \mu \circ \mu T$,

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\Downarrow\tau"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

- invertible modifications $\lambda : \mu \circ T\eta \to id_T$ and $\rho : \mu \circ \eta T \to id_T$,

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "id_T"'] & T^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Downarrow\lambda"] \\
 & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 \arrow[d, "\mu"'] \arrow[dr, phantom, "\Downarrow\rho"] & T \arrow[l, "\eta T"'] \arrow[dl, "id_T"] \\
T &
\end{tikzcd}$$

subject to the two coherence axioms below:

$$\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, "T\mu T"] & T^3 \arrow[dr, "T\mu"] \arrow[d, phantom, "\Downarrow T\tau"] & \\
T^3 \arrow[r, phantom, "\Downarrow\tau T"] \arrow[dr, "\mu T"'] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Downarrow\tau"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, phantom, "\Downarrow\mu_\mu"] & T^3 \arrow[d, "\mu T"] \arrow[dr, "T\mu"] & \\
T^3 \arrow[r, "T\mu"] \arrow[dr, "\mu T"'] & T^2 \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, phantom, "\Downarrow\tau"] \arrow[dr, "\mu"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "id_{T^2}"'] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dl, phantom, "\Downarrow\lambda T"] \arrow[dr, phantom, "\Downarrow\tau"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "id_{T^2}"'] & T^3 \arrow[d, "T\mu"] \arrow[dl, phantom, "\Downarrow T\rho"] & \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

We also need the notion of the monad morphism for pseudo-monads:

Definition 2.22 (pseudo-monad morphism). Given pseudo-monads $(T,\mu,\eta,\tau,\lambda,\rho)$ and $(T',\mu',\eta',\tau',\lambda',\rho')$ on a 2-category $\mathbb{C}$, a *pseudo-monad morphism* α from T to T' is a pseudo-natural transformation $\alpha : T \to T'$, together with two invertible modifications

$$
\begin{tikzcd}
T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu"'] \arrow[drr, phantom, "\Downarrow\overline{\alpha}_\mu"] & TT' \arrow[r, "\alpha T'"] & T'^2 \arrow[d, "\mu'"] \\
T \arrow[rr, "\alpha"'] & & T'
\end{tikzcd}
\qquad
\begin{tikzcd}
Id \arrow[r, "\eta"] \arrow[dr, "\eta'"'] \arrow[dr, phantom, shift left=3, "\Downarrow\overline{\alpha}_\eta"] & T \arrow[d, "\alpha"] \\
& T'
\end{tikzcd}
$$

subject to the following three coherence axioms:

$$
\begin{tikzcd}
& T^3 \arrow[r, "T^2\alpha"] \arrow[d, "T\mu"] \arrow[dl, "\mu T"'] \arrow[drr, phantom, "\Downarrow T\overline{\alpha}_\mu"] & T^2T' \arrow[r, "T\alpha T'"] & TT'^2 \arrow[r, "\alpha T'^2"] \arrow[d, "T\mu'"'] \arrow[dr, phantom, "\cong_\alpha"] & T'^3 \arrow[d, "T'\mu'"] \\
T^2 \arrow[r, phantom, "\Downarrow\tau"] \arrow[dr, "\mu"'] & T^2 \arrow[rr, "T\alpha"] \arrow[d, "\mu"] \arrow[drrr, phantom, "\Downarrow\overline{\alpha}_\mu"] & & TT' \arrow[r, "\alpha T'"] & T'^2 \arrow[d, "\mu'"] \\
& T \arrow[rrr, "\alpha"] & & & T'
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
T^3 \arrow[r, "T^2\alpha"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\cong_\mu"] & T^2T' \arrow[r, "T\alpha T'"] \arrow[d, "\mu T'"] \arrow[drr, phantom, "\Downarrow\overline{\alpha}_\mu T'"] & TT'^2 \arrow[r, "\alpha T'^2"] & T'^3 \arrow[d, "\mu' T'"'] \arrow[dr, "T'\mu'"] & \\
T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu"'] \arrow[drrr, phantom, "\Downarrow\overline{\alpha}_\mu"] & TT' \arrow[rr, "\alpha T'"] & & T'^2 \arrow[r, phantom, "\Downarrow\tau'"] \arrow[d, "\mu'"'] & T'^2 \arrow[dl, "\mu'"] \\
T \arrow[rrr, "\alpha"'] & & & T' &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu"] \arrow[drr, phantom, "\Downarrow\overline{\alpha}_\mu"] & TT' \arrow[r, "\alpha T'"] & T'^2 \arrow[d, "\mu'"] \\
T \arrow[ur, "T\eta"] \arrow[r, "id"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & T \arrow[rr, "\alpha"'] & & T'
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[r, "T\alpha"] \arrow[d, "T\eta"'] \arrow[dr, phantom, "\Downarrow T\overline{\alpha}_\eta"] & TT' \arrow[r, "\alpha T'"] \arrow[dr, phantom, "\cong_\alpha"] & T'^2 \arrow[dr, "\mu'"] & \\
T \arrow[ur, "T\eta'"'] \arrow[rr, "\alpha"'] & & T' \arrow[u, "T'\eta'"] \arrow[r, "id"'] \arrow[ur, phantom, shift left=3, "\Downarrow\lambda'"] & T'
\end{tikzcd}
$$

$$
\begin{tikzcd}
& T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu"] \arrow[drr, phantom, "\Downarrow\overline{\alpha}_\mu"] & TT' \arrow[r, "\alpha T'"] & T'^2 \arrow[d, "\mu'"] \\
T \arrow[ur, "\eta T"] \arrow[r, "id"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & T \arrow[rr, "\alpha"'] & & T'
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[r, "T\alpha"] \arrow[dr, phantom, "\cong_\eta"] & TT' \arrow[r, "\alpha T'"] \arrow[dr, phantom, "\Uparrow\overline{\alpha}_\eta T'"] & T'^2 \arrow[dr, "\mu'"] & \\
T \arrow[u, "\eta T"] \arrow[rr, "\alpha"'] & & T' \arrow[ul, "\eta T'"] \arrow[u, "\eta' T'"'] \arrow[r, "id"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho'"] & T'
\end{tikzcd}
$$

2.4 The 2-category of pseudo-T-algebras

Now we consider algebras of pseudo-monads, in the pseudo-setting.

Definition 2.23 (Pseudo-T-algebra). Given a pseudo-monad $(T,\mu,\eta,\tau,\lambda,\rho)$ on a 2-category $\mathbb{C}$, a *pseudo-T-algebra* $\langle A,a,a_\mu,a_\eta\rangle$ consists of the following data:

- a 0-cell A of $\mathbb{C}$
- a 1-cell $a : TA \to A$
- invertible 2-cells $a_\mu : a\circ Ta \to a\circ \mu_A$, $a_\eta : a\circ\eta_A \to id_A$

$$
\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow a_\mu"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] \arrow[dr, phantom, shift left=3, "\Downarrow a_\eta"] & TA \arrow[d, "a"] \\
 & A
\end{tikzcd}
$$

subject to the following coherence axioms: for the associative law,

$$
\begin{tikzcd}
T^3A \arrow[r, "T^2a"] \arrow[d, "\mu_{TA}"'] \arrow[dr, "T\mu_A"] & T^2A \arrow[dr, "Ta"] \arrow[d, phantom, "\Downarrow Ta_\mu"] & \\
T^2A \arrow[r, phantom, "\Downarrow \tau_A"] \arrow[dr, "\mu_A"'] & T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\Downarrow a_\mu"] & TA \arrow[d, "a"] \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
T^3A \arrow[r, "T^2a"] \arrow[d, "\mu_{TA}"'] \arrow[dr, phantom, "\Downarrow \mu_a"] & T^2A \arrow[d, "\mu_A"] \arrow[dr, "Ta"] & \\
T^2A \arrow[r, "Ta"] \arrow[dr, "\mu_A"'] \arrow[dr, phantom, shift left=3, "\Downarrow a_\mu"] & TA \arrow[r, phantom, "\Downarrow a_\mu"] \arrow[dr, "a"] & TA \arrow[d, "a"] \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
$$

and for the left unit law,

$$
\begin{tikzcd}
TA \arrow[r, "T\eta_A"] \arrow[dr, "id_{TA}"'] \arrow[dr, phantom, shift left=3, "\Downarrow \lambda_A"] & T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\Downarrow a_\mu"] & TA \arrow[d, "a"] \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "T\eta_A"] \arrow[dr, "id_{TA}"'] \arrow[dr, phantom, shift left=3, "\Downarrow Ta_\eta"] & T^2A \arrow[d, "Ta"] & \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
$$

and from these two axioms follow another axiom for the unit law:

$$
\begin{tikzcd}
TA \arrow[r, "\eta_{TA}"] \arrow[dr, "id_{TA}"'] \arrow[dr, phantom, "\Downarrow\rho_A", near start] & T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow a_\mu"] & TA \arrow[d, "a"] \\
 & TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\eta_{TA}"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\eta_a"] & T^2A \arrow[d, "Ta"] \\
A \arrow[r, "\eta_A"] \arrow[dr, "id_{TA}"'] \arrow[dr, phantom, "\Downarrow Ta_\eta", near start] & TA \arrow[d, "a"] \\
 & A
\end{tikzcd}
$$

Definition 2.24 (Pseudo-map). A *pseudo-map* $(f, \overline{f}_{a,b})$ of pseudo-T-algebras from $\langle A, a, a_\mu, a_\eta\rangle$ to $\langle B, b, b_\mu, b_\eta\rangle$ consists of a 1-cell $f : A \to B$ and an invertible 2-cell $\overline{f}_{a,b} : b \circ Tf \to f \circ a$

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}_{a,b}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

subject to two coherence axioms:

$$
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] \arrow[dr, "Ta"] \arrow[dr, phantom, "\Downarrow T\overline{f}_{a,b}", shift left=4] & T^2B \arrow[dr, "Tb"] & \\
TA \arrow[dr, "a"'] \arrow[r, phantom, "\Downarrow a_\mu"] & TA \arrow[r, "Tf"] \arrow[d, "a"] \arrow[dr, phantom, "\Downarrow\overline{f}_{a,b}"] & TB \arrow[d, "b"] \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow\mu_f"] & T^2B \arrow[d, "\mu_B"] \arrow[dr, "Tb"] & \\
TA \arrow[r, "Tf"] \arrow[dr, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}_{a,b}", shift left=4] & TB \arrow[dr, "b"'] \arrow[r, phantom, "\Downarrow b_\mu"] & TB \arrow[d, "b"] \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "\eta_A"] \arrow[ddr, "id_A"'] \arrow[dr, phantom, "\Downarrow\eta_f", shift left=4] \arrow[ddr, phantom, "\Downarrow a_\eta", shift left=3] & B \arrow[dr, "\eta_B"] & \\
 & TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}_{a,b}"] & TB \arrow[d, "b"] \\
 & A \arrow[r, "f"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[dr, "\eta_B"] \arrow[ddr, "id_B"'] \arrow[ddr, phantom, "\Downarrow b_\eta", shift left=3] & \\
 & & TB \arrow[d, "b"] \\
 & & B
\end{tikzcd}
$$

Definition 2.25. An *algebra 2-cell* from $(f,\overline{f}_{a,b})$ to $(g,\overline{g}_{a,b})$ is a 2-cell $\chi : f \Rightarrow g$ subject to the following coherence axiom:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & TB \arrow[d, "b"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \chi"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T\chi"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{g}_{a,b}"] & TB \arrow[d, "b"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

2.4.1 The 2-category *Ps-T-Alg*

Definition 2.26 (the 2-category of pseudo-algebras). The above definitions together form a 2-category of pseudo-T-algebras, *Ps-T-Alg*, where the 0-cells are pseudo-T-algebras, the 1-cells are pseudo-maps of pseudo-T-algebras, and the 2-cells are algebra 2-cells. The composition functor is defined as follows: for pseudo-T-algebras $\langle A, a, a_\mu, a_\eta \rangle$ and $\langle B, b, b_\mu, b_\eta \rangle$, the composition functor is given as

$$comp_{A,B,C} : \textit{Ps-T-Alg}(\langle B,b\rangle, \langle C,c\rangle) \times \textit{Ps-T-Alg}(\langle A,a\rangle, \langle B,b\rangle) \to \textit{Ps-T-Alg}(\langle A,a\rangle, \langle C,c\rangle)$$

which sends a pair of 1-cells, $(f,\overline{f}_{a,b}) : \langle A, a, a_\mu, a_\eta \rangle \to \langle B, b, b_\mu, b_\eta \rangle$ and $(g,\overline{g}_{b,c}) : \langle B, b, b_\mu, b_\eta \rangle \to \langle C, c, c_\mu, c_\eta \rangle$, to $(gf, \overline{gf}_{a,c}) : \langle A, a, a_\mu, a_\eta \rangle \to \langle C, c, c_\mu, c_\eta \rangle$, where gf is the composite of 1-cells in $\mathbb{C}$ and $\overline{gf}_{a,c}$ is defined as the composite of invertible 2-cells, $\overline{gf}_{a,c} = (\overline{g}_{b,c} \circ Tf) \cdot (g \circ \overline{f}_{a,b})$, as shown below:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \overline{f}_{a,b}"] & TB \arrow[r, "Tg"] \arrow[d, "b"] \arrow[dr, phantom, "\Downarrow \overline{g}_{b,c}"] & TC \arrow[d, "c"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

From this it is easy to see that $(gf, \overline{gf}_{a,c})$ satisfies the axioms for pseudo-maps and that this is a well-defined definition. The identity in *Ps-T-Alg*$(\langle A,a\rangle, \langle A,a\rangle)$ is $(id_A, \overline{id}_{a,a})$. The functor *comp* defines the composition of 2-cells as the horizontal composition, which obviously preserves pseudo-maps.

2.5 Bicategories and bimonoidal bicategories

Definition 2.27 (bicategory). A *bicategory* $\mathbb{C}$ consists of the data for a 2-category (Definition 2.12) , i.e., 0-cells, 1-cells, and 2-cells together with families of functors *comp* and *unit*, with the commutativity constraints for those functors replaced by the existence of some natural isomorphisms whose components (invertible 2-cells) are described in the following diagrams:

$$\begin{tikzcd}
\mathbb{C}(C,D)\times\mathbb{C}(B,C)\times\mathbb{C}(A,B) \arrow[r, "comp_{B,C,D}\times id_{\mathbb{C}(A,B)}"] \arrow[d, "id_{\mathbb{C}(C,D)}\times comp_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow\alpha_{A,B,C,D}"] & \mathbb{C}(B,D)\times\mathbb{C}(A,B) \arrow[d, "comp_{A,B,D}"] \\
\mathbb{C}(C,D)\times\mathbb{C}(A,C) \arrow[r, "comp_{A,C,D}"] & \mathbb{C}(A,D)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}(A,B) \arrow[r, "id_{\mathbb{C}(A,B)}\times unit_A"] \arrow[d, "unit_B\times id_{\mathbb{C}(A,B)}"'] \arrow[dr, "id_{\mathbb{C}(A,B)}"] \arrow[d, phantom, shift left=8ex, "\Uparrow\lambda_{A,B}"] & \mathbb{C}(A,B)\times\mathbb{C}(A,A) \arrow[d, "comp_{A,A,B}"] \arrow[d, phantom, shift right=8ex, "\Downarrow\rho_{A,B}"] \\
\mathbb{C}(B,B)\times\mathbb{C}(A,B) \arrow[r, "comp_{A,B,B}"'] & \mathbb{C}(A,B)
\end{tikzcd}$$

subject to the following two coherence axioms: suppressing the subscripts for the components and using $\circ$ instead of *comp*, for a composable quadruple of 1-cells f,g,h and k, they are expressed as commutative diagrams:

$$\begin{tikzcd}
k\circ(h\circ(g\circ f)) \arrow[r, "\alpha"] \arrow[d, "id_k\circ\alpha"'] & (k\circ h)\circ(g\circ f) \arrow[r, "\alpha"] & ((k\circ h)\circ g)\circ f \\
k\circ((h\circ g)\circ f) \arrow[rr, "\alpha"'] & & (k\circ(h\circ g))\circ f \arrow[u, "\alpha\circ id_f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
g\circ(id_B\circ f) \arrow[rr, "\alpha"] \arrow[dr, "id_g\circ\lambda"'] & & (g\circ id_B)\circ f \arrow[dl, "\rho\circ id_f"] \\
& g\circ f &
\end{tikzcd}$$

Evidently, a bicategory is a 2-category if all the invertible 2-cells described above are identities. Also note the similarity of the coherence axioms to the commutativity axioms for monoidal categories defined in Definition 2.8. This reflects the well known fact that a monoidal category is regarded as a one object (0-cell) bicategory.

In Chapter 6, we need the notion of bimonoidal 2-category, which can be conceptually described as a 2-category with a tensor given by a pseudo-functor. In fact, the above identification of a monoidal category with a bicategory extends to the level of 3-category, i.e., a tricategory with one object is a bimonoidal bicategory. In the following, we give the relevant definition needed for the discussion in Chapter 6.

Definition 2.28 (bimonoidal bicategory, [GPS95]). A *bimonoidal bicategory* $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of the following data:

- a bicategory $\mathbb{C}$,
- a pseudo-functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, called the *tensor*,
- an object $I \in \mathbb{C}$, called the *unit*,
- three pseudo-natural isomorphisms

$$\alpha : - \otimes (- \otimes -) \cong (- \otimes -) \otimes -$$
$$\lambda : I \otimes - \cong Id_{\mathbb{C}}$$
$$\rho : - \otimes I \cong Id_{\mathbb{C}}$$

- four invertible modifications as described below:

$$\begin{tikzcd}
- \otimes (- \otimes (- \otimes -)) \arrow[r, "\alpha"] \arrow[d, "id \otimes \alpha"'] & (- \otimes -) \otimes (- \otimes -) \arrow[r, "\alpha"] \arrow[d, phantom, "\Uparrow \overline{m}^1"] & ((- \otimes -) \otimes -) \otimes - \\
- \otimes ((- \otimes -) \otimes -) \arrow[rr, "\alpha"'] & & (- \otimes (- \otimes -)) \otimes - \arrow[u, "\alpha \otimes id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
- \otimes (I \otimes -) \arrow[r, "\alpha"] \arrow[dr, "id \oplus \lambda"'] \arrow[r, phantom, shift right=4, "\Downarrow \overline{m}^2"] & (- \otimes I) \otimes - \arrow[d, "\rho \otimes id"] \\
 & - \otimes -
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes (- \otimes -) \arrow[r, "\alpha"] \arrow[dr, "\lambda"'] \arrow[r, phantom, shift right=4, "\Downarrow \overline{m}^3"] & (I \otimes -) \otimes - \arrow[d, "\lambda \otimes id"] \\
 & - \otimes -
\end{tikzcd}
\qquad
\begin{tikzcd}
- \otimes (- \otimes I) \arrow[r, "\alpha"] \arrow[dr, "id \oplus \rho"'] \arrow[r, phantom, shift right=4, "\Downarrow \overline{m}^4"] & (- \otimes -) \otimes I \arrow[d, "\rho"] \\
 & - \otimes -
\end{tikzcd}$$

The four invertible modifications are subject to the three coherence axioms given below:

$$\begin{tikzcd}
& -\otimes(-\otimes-) \arrow[r, "\alpha"] \arrow[dr, phantom, "\cong_\alpha"] & (-\otimes-)\otimes- \\
-\otimes(I\otimes(-\otimes-)) \arrow[ur, "id\otimes\lambda"] \arrow[r, "\alpha"'] \arrow[d, "id\otimes\alpha"'] \arrow[ur, phantom, "\Uparrow\overline{m}^2"', near end] & (-\otimes I)\otimes(-\otimes-) \arrow[u, "\rho\otimes id"'] \arrow[r, "\alpha"'] \arrow[d, phantom, "\Uparrow\overline{m}^1"] & ((-\otimes I)\otimes-)\otimes- \arrow[u, "(\rho\otimes id)\otimes id"'] \\
-\otimes((I\otimes-)\otimes-) \arrow[rr, "\alpha"] & & (-\otimes(I\otimes-))\otimes- \arrow[u, "\alpha\otimes id"']
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
-\otimes(I\otimes(-\otimes-)) \arrow[r, "id\otimes\lambda"] \arrow[dd, "id\otimes\alpha"'] & -\otimes(-\otimes-) \arrow[r, "\alpha"] & (-\otimes-)\otimes- & \\
\arrow[r, phantom, "\Uparrow id\otimes\overline{m}^3"] & \arrow[r, phantom, "\cong_\alpha"] & \arrow[r, phantom, "\overset{\Leftarrow}{\overline{m}^2}\otimes id"] & ((-\otimes I)\otimes-)\otimes- \arrow[ul, "(\rho\oplus id)\oplus id"'] \\
-\otimes((I\otimes-)\otimes-) \arrow[uur, "id\oplus(\lambda\oplus id)"] \arrow[rr, "\alpha"] & & (-\otimes(I\otimes-))\otimes- \arrow[uu, "(id\otimes\lambda)\otimes id"] \arrow[ur, "\alpha\oplus id"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& -\otimes(-\otimes-) \arrow[r, "\alpha"] \arrow[d, phantom, "\cong_\alpha"] & (-\otimes-)\otimes- \\
-\otimes(-\otimes(I\otimes-)) \arrow[ur, "id\otimes(id\otimes\lambda)"] \arrow[r, "\alpha"'] \arrow[d, "id\otimes\alpha"'] & (-\otimes-)\otimes(I\otimes-) \arrow[ur, "id\otimes\lambda"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\Uparrow\overline{m}^2"', near end] \arrow[d, phantom, "\Uparrow\overline{m}^1"] & ((-\otimes-)\otimes I)\otimes- \arrow[u, "\rho\otimes id"'] \\
-\otimes((-\otimes I)\otimes-) \arrow[rr, "\alpha"] & & (-\otimes(-\otimes I))\otimes- \arrow[u, "\alpha\otimes id"']
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
-\otimes(-\otimes(I\otimes-)) \arrow[r, "id\otimes(id\otimes\lambda)"] \arrow[dd, "id\otimes\alpha"'] & -\otimes(-\otimes-) \arrow[r, "\alpha"] & (-\otimes-)\otimes- & \\
\arrow[r, phantom, "\Uparrow id\otimes\overline{m}^2"] & \arrow[r, phantom, "\cong_\alpha"] & \arrow[r, phantom, "\overset{\Leftarrow}{\overline{m}^4}\otimes id"] & ((-\otimes-)\otimes I)\otimes- \arrow[ul, "\rho\oplus id"'] \\
-\otimes((-\otimes I)\otimes-) \arrow[uur, "id\oplus(\rho\oplus id)"] \arrow[rr, "\alpha"] & & (-\otimes(-\otimes I))\otimes- \arrow[uu, "(id\otimes\rho)\otimes id"] \arrow[ur, "\alpha\oplus id"'] &
\end{tikzcd}$$

A *strong bimonoidal bifunctor* is defined to be exactly a trifunctor of one object tricategories [GPS95], and that means that all the structure of the bimonoidal bicategory is preserved up to coherent equivalence, the coherence axioms corresponding exactly to the pentagon and triangle in the definition of monoidal category [GPS95].

Chapter 3

Distributive Laws

In this chapter we study distributive laws of a monad over an endofunctor and a natural transformation and also over a monad. Given a monad (S,μ,η) and an endofunctor H on a category $\mathcal{C}$, a distributive law δ is a natural transformation $\delta : SH \to HS$ that satisfies two axioms. The main theorem of the chapter is that the existence of such a distributive law induces a lifting of H to the category of S-algebras, S-*Alg*, and vice versa. We then consider the case where H is a monad.

The results presented in this chapter are known in one way or another from the literature (for example [BW85]), but it is worthwhile for the development of the thesis to spell them out here in detail, because it will offer great guidance through the discussion of pseudo-distributive laws in the following chapters, where the axioms in this chapter systematically become data (invertible 2-cells), and the proofs become constructions.

This chapter comes in two parts: in the first five sections we study the relationship between a distributive law of a monad S over an endofunctor and a lifting of the endofunctor to the category of S-algebras S-*Alg*, and that between distributivity of a monad over a natural transformation and a lifting of the natural transformation to S-*Alg*. In Section 3.1 we give the definition of a distributive law of a monad S over an endofunctor H, followed by the definition of the notion of the distributivity of S over a natural transformation. In Section 3.2, we define a lifting of an endofunctor H to the category of S-algebras, S-*Alg*. A lifting of H to S-*Alg* is an endofunctor $\widehat{H}$ such that $U\widehat{H} = HU$ holds for the forgetful functor $U : S\text{-}Alg \to \mathcal{C}$. This means $\widehat{H}$ sends an S-algebra $\langle A, a\rangle$ to $\langle HA, \widehat{a}\rangle$, where $\widehat{a}$ is the structure map from SHA to HA, and an S-algebra map f

to Hf, which is again an S-algebra map. We then describe the condition for a natural transformation between endofunctors with liftings to S-Alg also to lift. These definitions yield the categories of distributive laws of S over endofunctors, called $\mathsf{Dist}^{\mathsf{S}}$, and of liftings of endofunctors to S-Alg, called $\mathsf{Lift}_{S\text{-}Alg}$.

In the following two sections we go on to prove that a distributive law of S over an endofunctor induces a lifting of that endofunctor to S-Alg, and vice versa. We do similarly for natural transformations. We rephrase our discussion in terms of categories and prove that $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{S\text{-}Alg}$ are isomorphic in Section 3.5. Note that this isomorphism becomes an equivalence in the pseudo-case as we see in Chapter 7.

Next, in Section 3.6, we state a few propositions that pave our way for the study of distributive laws over a monad in Section 3.7. The goal of the discussion here is to establish the relation between the composite $\widehat{H} \circ \widehat{H}$ and the distributive law of the form $H\delta \circ \delta H$. We prove that the first is a lifting of H^2 and the latter is a distributive law of S over H^2 and that the isomorphism of the categories described in the previous section sends one to the other. This result is essential in the discussion in the following section because the definition of lifting of a monad T to a monad $\widehat{T}$ on S-Alg requires that both the multiplication μ and the unit η lift to S-Alg. The discussion in this section will be elaborated in Chapter 6, extending the discussion further to the pseudo case.

In the second part of the chapter, we consider the situation where the endofunctor carries the structure of a monad: distributive laws of S over monads are those over endofunctors with two additional axioms which follow from the compatibility with the multiplication and the unit of monads. We also see how our definition of liftings extends to those of monads, the main point of which is the lifting of natural transformations μ and η. The definition of distributive laws over a monad is given in Section 3.7 and that of liftings of a monad to a monad on S-Alg in Section 3.7.1. Then we establish the relation between such distributive laws and such liftings, which follows from the isomorphism for the case with endofunctors. We prove that the isomorphism of categories for endofunctors preserves the monad properties: they induce functors between the categories, $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$ and $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$, that define an isomorphism between them.

In the last section, Section 3.8, we study several properties given two monads S and T and a distributive law $ST \to TS$ of S over T. First we prove that the composite functor TS acquires the structure of a monad induced by the distributive law. Then we investigate the structure of algebras of this composite monad TS, and compare those with the algebras of the lifting $\widehat{T}$ of T to S-Alg. We show that the comparison functor between the Eilenberg-Moore adjunction defined by the monad TS and that of the composite of those defined by the monads S and $\widehat{T}$ is isomorphic, proving that the categories TS-Alg and $\widehat{T}$-Alg are canonically isomorphic. These results are extended to the pseudo-case in Section 7.4.

3.1 Distributivity of a monad *S*

We start our discussion with the definition of a distributive law of a monad over an endofunctor. Such a distributive law consists of a natural transformation satisfying two conditions, whereas we will need two extra conditions when we extend it to a distributive law of a monad over a monad later in this chapter. When we move onto the discussion of pseudo-distributive laws, the axioms here become data, i.e., invertible 2-cells that satisfy several coherence conditions.

Definition 3.1. Given a monad (S,μ,η) and an endofunctor H on a category $\mathcal{C}$, a *distributive law of S over H* is a natural transformation

$$\delta : SH \longrightarrow HS$$

which makes the following diagrams commute:

$$\begin{tikzcd} S^2H \arrow[r, "S\delta"] \arrow[d, "\mu H"'] & SHS \arrow[r, "\delta S"] & HS^2 \arrow[d, "H\mu"] \\ SH \arrow[rr, "\delta"'] & & HS \end{tikzcd} \tag{δ-μ}$$

$$\begin{tikzcd} H \arrow[d, "\eta H"'] \arrow[dr, "H\eta"] & \\ SH \arrow[r, "\delta"'] & HS \end{tikzcd} \tag{δ-η}$$

We call the first diagram the associative law for distributive laws, or δ-μ, and the second the unit law for distributive laws, or δ-η.

In the presence of distributive laws of S over endofunctors H and K, a natural transformation $\alpha : H \to K$ with a certain property can be regarded as a transformation of these distributive laws, or to put it differently, "S distributes over α" with respect to these laws.

Definition 3.2. Given distributive laws $\delta^H : SH \to HS$ and $\delta^K : SK \to KS$ and a natural transformation $\alpha : H \to K$, we say *S distributes over α with respect to δ^H and δ^K* if

$$\begin{tikzcd} SH \arrow[r, "\delta^H"] \arrow[d, "S\alpha"'] & HS \arrow[d, "\alpha S"] \\ SK \arrow[r, "\delta^K"'] & KS \end{tikzcd} \tag{3.1}$$

holds.

The distributive laws of S over endofunctors and the natural transformations that S distributes over as defined above form a category.

Proposition 3.3. *The data defined above form a category we denote by Dist^S as follows: objects of Dist^S are pairs $(H, \delta : SH \to HS)$ of an endofunctor H on $\mathcal{C}$ and a distributive law of S over it, and an arrow from (H, δ^H) and (K, δ^K) is given by a natural transformation $\alpha : H \to K$ that S distributes over with respect to δ^H and δ^K.*

Proof. The composition of arrows is given by composition of natural transformations (3.1). The rest follows by routine calculation. $\square$

Notation 3.4. We often omit the first component in the objects whenever it does not cause confusion and just write δ^H instead of $(H, \delta : SH \to HS)$.

3.2 Lifting to *S-Alg*

Given a monad (S, μ, η) and an endofunctor H on a category $\mathcal{C}$, we define the notion of a lifting of H to the category *S-Alg*.

Definition 3.5. A *lifting* of H to *S-Alg* is an endofunctor $\widehat{H}$ on *S-Alg* for which $U\widehat{H} = HU$ holds, where U is the forgetful functor from *S-Alg* to $\mathcal{C}$.

Hence $\widehat{H}$ is an endofunctor on *S-Alg* such that for an S-algebra $\langle A, a\rangle$ and a map of S-algebras $f : \langle A, a\rangle \to \langle B, b\rangle$, we have

$$U\widehat{H}\langle A, h\rangle = HU\langle A, h\rangle = HA \tag{3.2a}$$

$$U\widehat{H} f = HU f = H f. \tag{3.2b}$$

From (3.2a) we know that $\widehat{H}\langle A, a\rangle$ consists of an S-algebra structure on HA, hence in the following we write $\widehat{H}\langle A, a\rangle = \langle HA, \widehat{a}\rangle$ where the structure map $\widehat{a} : SHA \to HA$ should satisfy the following commuting diagrams.

$$\begin{tikzcd} S^2HA \arrow[r, "S\widehat{a}"] \arrow[d, "\mu_{HA}"'] & SHA \arrow[d, "\widehat{a}"] \\ SHA \arrow[r, "\widehat{a}"'] & HA \end{tikzcd} \qquad \begin{tikzcd} HA \arrow[r, "\eta_{HA}"] \arrow[dr, "id_{HA}"'] & SHA \arrow[d, "\widehat{a}"] \\ & HA \end{tikzcd} \tag{3.3a}$$

For the arrow part, (3.2b) states that $Hf : \widehat{H}\langle A, a\rangle \to \widehat{H}\langle B, b\rangle$ is an S-algebra map, so the diagram below commutes:

$$\begin{tikzcd} SHA \arrow[r, "SHf"] \arrow[d, "\widehat{a}"'] & SHB \arrow[d, "\widehat{b}"] \\ HA \arrow[r, "Hf"'] & HB \end{tikzcd} \tag{3.3b}$$

Notation 3.6. We write $\widehat{a}^{\widehat{H}}$ to denote the structure map of the value of $\widehat{H}$ at $\langle A, a\rangle$, whenever necessary to make it clear which lifting is concerned. Otherwise we will simply write $\widehat{a}$.

Given liftings $\widehat{H}$ and $\widehat{K}$ of H and K, respectively, a natural transformation $\alpha : H \to K$ with a certain property is also a natural transformation from $\widehat{H}$ to $\widehat{K}$.

Definition 3.7. Given endofunctors H and K on $\mathcal{C}$ with their liftings $\widehat{H}$ and $\widehat{K}$ and a natural transformation $\alpha : H \to K$, we say "α *lifts to S-Alg from* $\widehat{H}$ *to* $\widehat{K}$" if, for any

S-algebra $\langle A, a \rangle$, α_A is an S-algebra map from $\widehat{H}\langle A, a \rangle$ to $\widehat{K}\langle A, a \rangle$, or equivalently, α_A makes the diagram

$$\begin{tikzcd} SHA \arrow[r, "S\alpha_A"] \arrow[d, "\widehat{a}^H"'] & SKA \arrow[d, "\widehat{a}^K"] \\ HA \arrow[r, "\alpha_A"'] & KA \end{tikzcd} \tag{3.4}$$

commute.

In the following, we also use the notation $\widehat{\alpha} : \widehat{H} \to \widehat{K}$ to refer to α regarded as a natural transformation between liftings and call it a *lifting of* α *to S-Alg*.

All the liftings of endofunctors on $\mathcal{C}$ to *S-Alg* and the natural transformations between endofunctors on $\mathcal{C}$ that lift to *S-Alg* as defined above form the category $\mathsf{Lift}_{S\text{-}Alg}$ as follows:

Proposition 3.8. *The data defined above form a category we denote by* $\mathsf{Lift}_{S\text{-}Alg}$*: objects of* $\mathsf{Lift}_{S\text{-}Alg}$ *are pairs* $(H, \widehat{H})$ *of an endofunctor* H *on* $\mathcal{C}$ *and an endofunctor* $\widehat{H}$ *on S-Alg such that* $U\widehat{H} = HU$ *holds, and an arrow from* $(H, \widehat{H})$ *to* $(K, \widehat{K})$ *is given by a natural transformation* $\alpha : H \to K$ *that lifts to S-Alg from* $\widehat{H}$ *to* $\widehat{K}$*.*

Proof. Follows by routine calculation. The composition of arrows is given by the composition in $\mathbb{C}$. ☐

Notation 3.9. We omit the first component in the objects whenever it does not cause confusion and just write $\widehat{H}$ instead of $(H, \widehat{H})$.

3.3 From liftings to distributive laws

Before moving on to the discussion of distributive laws over monads, in the next few sections we prove the isomorphism between distributive laws over endofunctors and liftings of endofunctors to *S-Alg*. We proceed by first providing the proofs that each of them induces the other, and then proving that the correspondences in those proofs define an isomorphism of categories between the category of distributive laws and that of liftings.

We start by proving that a lifting of H to S-Alg induces a distributive law of S over H, then we define a functor from $\mathsf{Lift}_{S\text{-}Alg}$ to $\mathsf{Dist}^{\mathsf{S}}$. To that end, we need to study some properties of free S-algebras.

Recall that, given any monad (S,μ,η), the component μ_A of the multiplication always yields an S-algebra $\langle SA,\mu_A\rangle$, the free S-algebra on A. Now, consider the value at $\langle SA,\mu_A\rangle$ of a lifting $\widehat{H}$ of H for each A. We observe the following:

Lemma 3.10. *The collection $\{\widehat{\mu}_A\}_A$ of the structure maps of $\widehat{H}\langle SA,\mu_A\rangle$ for each object A of $\mathcal{C}$ is natural in A, that is, it defines a natural transformation $\widehat{\mu} : SHS \to HS$.*

Proof. First note that, for any arrow $f : A \to B$, the arrow $Sf : SA \to SB$ is an S-algebra map from $\langle SA,\mu_A\rangle$ to $\langle SB,\mu_B\rangle$, so is sent by $\widehat{H}$ to an S-algebra map from $\langle HSA,\widehat{\mu}_A\rangle$ to $\langle HSB,\widehat{\mu}_B\rangle$. Then one can obtain the naturality square for f immediately by applying the diagram (3.3b) to Sf with $a=\mu_A$ and $b=\mu_B$:

$$\begin{tikzcd} SHSA \arrow[r, "SHSf"] \arrow[d, "\widehat{\mu}_A"'] & SHSB \arrow[d, "\widehat{\mu}_B"] \\ HSA \arrow[r, "HSf"'] & HSB \end{tikzcd}$$

$\square$

Our next observation is that the structure map a of any S-algebra $\langle A,a\rangle$ is always an S-algebra map from μ_A to a:

Lemma 3.11. *For any S-algebra $\langle A,a\rangle$, the structure map $a : SA \to A$ is an S-algebra map from $\langle SA,\mu_A\rangle$ to $\langle A,a\rangle$.*

Proof. Follows from the associative law for S-algebras. $\square$

Since this a is sent by $\widehat{H}$ to the S-algebra map $Ha : \widehat{H}\langle SA,\mu_A\rangle \to \widehat{H}\langle A,a\rangle$, the following diagram commutes for any $a : SA \to A$:

$$\begin{tikzcd} SHSA \arrow[r, "SHa"] \arrow[d, "\widehat{\mu}_A"'] & SHA \arrow[d, "\widehat{a}"] \\ HSA \arrow[r, "Ha"'] & HA \end{tikzcd} \tag{3.5}$$

Now we are ready to prove the next proposition: we construct a natural transformation using $\widehat{\mu}$ discussed above and prove that it satisfies the conditions to be a distributive law. In the later chapters when we discuss pseudo-distributive laws, the commutative squares are replaced by invertible 2-cells, and the proofs become constructions.

Proposition 3.12. *Given a monad (S, μ, η) and an endofunctor H on a category $\mathcal{C}$, a lifting $\widehat{H}$ of H to S-Alg gives rise to a distributive law of S over H.*

Proof. From Lemma 3.10, we have the natural transformation $\widehat{\mu} : SHS \to HS$, whose component at A is the structure map $\widehat{\mu}_A$ of $\widehat{H}\langle SA, \mu_A\rangle$. Using this natural transformation $\widehat{\mu}$ we construct a distributive law $\Theta^H(\widehat{H})$ by letting

$$\Theta^H(\widehat{H}) = \widehat{\mu} \circ SH\eta : SH \xrightarrow{SH\eta} SHS \xrightarrow{\widehat{\mu}} HS. \tag{3.6}$$

It is immediate that this defines a natural transformation. Then it remains to prove that $\Theta^H(\widehat{H})$ satisfies the associative law (δ-μ) and the unit law (δ-η) for distributive laws. First, for the associative law of $\Theta^H(\widehat{H})$ and μ, the component at A is given as

$$\begin{tikzcd}
S^2HA \arrow[r, "S^2H\eta_A"] \arrow[dd, "\mu_{HA}"'] \arrow[ddr, phantom, "(1)"] & S^2HSA \arrow[r, "S\widehat{\mu}_A"] \arrow[ddr, "\mu_{HSA}"] & SHSA \arrow[r, "SH\eta_{SA}"] \arrow[dr, "id"'] \arrow[dd, phantom, "(2)"] & SHS^2A \arrow[r, "\widehat{\mu}_{SA}"] \arrow[d, "SH\mu_A"] & HS^2A \arrow[dd, "H\mu_A"] \\
 & & & SHSA \arrow[dr, "\widehat{\mu}_A"'] \arrow[r, phantom, "(4)"] & \\
SHA \arrow[rr, "SH\eta_A"'] & & SHSA \arrow[rr, "\widehat{\mu}_A"'] & & HSA
\end{tikzcd}$$

which commutes. The reason for commutativity of each area is given as follows: (1) commutes by the naturality of μ, (2) by the associative law for the S-algebra structure $\widehat{\mu}_A$, (3) by the right unit law for the monad S, and (4) by the diagram (3.5) with $a = \mu_A$.

For the unit law of $\Theta^H(\widehat{H})$ and η,

$$\begin{tikzcd}
HA \arrow[dr, "H\eta_A"] \arrow[dd, "\eta_{HA}"'] & & \\
(6) & HSA \arrow[d, "\eta_{HSA}"'] \arrow[dr, "id"] & \\
SH \arrow[r, "SH\eta_A"'] & SHSA \arrow[r, "\widehat{\mu}_A"'] & HSA
\end{tikzcd}$$

the square (6) commutes by the naturality of η, and the triangle (7) by the unit law for the S-algebra map $\widehat{\mu}_A$. This proves the proposition. $\square$

At the level of natural transformations, the proposition above induces the following correspondence between the arrows in $\mathsf{Lift}_{S\text{-}Alg}$ and those in $\mathsf{Dist}^{\mathsf{S}}$.

Proposition 3.13. *Given endofunctors H, K on a category $\mathcal{C}$, together with their liftings $\widehat{H}$, $\widehat{K}$ to S-Alg, and a natural transformation $\alpha : H \to K$, the following holds: if α lifts to S-Alg from $\widehat{H}$ to $\widehat{K}$ then S distributes over α with respect to the induced distributive laws $\Theta^H(\widehat{H})$ and $\Theta^K(\widehat{K})$.*

Proof. Assume that α lifts to *S-Alg* from $\widehat{H}$ to $\widehat{K}$, that is, for any S-algebra $\langle A, a\rangle$, the diagram (3.4) holds. Now recall that $\Theta^H(\widehat{H}) = \widehat{\mu}^{\widehat{H}} \circ SH\eta$ and $\Theta^K(\widehat{K}) = \widehat{\mu}^{\widehat{K}} \circ SK\eta$. What we need to show is that the diagram (3.1) commutes for these distributive laws:

$$\begin{tikzcd}
SHA \arrow[r, "SH\eta_A"] \arrow[d, "S\alpha_A"'] & SHSA \arrow[r, "\widehat{\mu}_A^{\widehat{H}}"] \arrow[d, "S\alpha_{SA}"] & HSA \arrow[d, "\alpha_{SA}"] \\
SKA \arrow[r, "SK\eta_A"'] & SKSA \arrow[r, "\widehat{\mu}_A^{\widehat{K}}"'] & KSA
\end{tikzcd}$$

This holds because of the diagram (3.4) for the S-algebra $\langle SA, \mu_A\rangle$ and the naturality of α. This proves the proposition. $\square$

This amounts to saying that if a natural transformation $\alpha : H \to K$ is an arrow in $\mathsf{Lift}_{S\text{-}Alg}$, it has to be an arrow in $\mathsf{Dist}^{\mathsf{S}}$ too. We now define a functor using the correspondence estabilished in Proposition 3.12 and Proposition 3.13.

Corollary 3.14. *The mapping Θ^H in Proposition 3.12 defines a faithful functor Θ from $\mathsf{Lift}_{S\text{-}Alg}$ to $\mathsf{Dist}^{\mathsf{S}}$.*

Proof. Define $\Theta(H, \widehat{H}) = \Theta^H(\widehat{H})$ and $\Theta(\alpha) = \alpha : \Theta(\widehat{H}) \to \Theta(\widehat{K})$ for any $\alpha : \widehat{H} \to \widehat{K}$. It is straightforward to verify its functoriality and faithfulness. $\square$

3.4 From distributive laws to liftings

Now we prove the opposite direction: the discussion proceeds similarly, by first proving that a distributive law induces a lifting, then after a little discussion on natural transformations, we finally define a faithful functor from $\mathsf{Dist}^{\mathsf{S}}$ to $\mathsf{Lift}_{S\text{-}Alg}$.

Proposition 3.15. *Given a monad (S,μ,η) and an endofunctor H on a category $\mathcal{C}$, a distributive law of S over H gives rise to a lifting of H to S-Alg.*

Proof. Given a distributive law $\delta : SH \to HS$, we construct an endofunctor $\Xi^H(\delta)$ on S-*Alg* as follows: for an S-algebra $\langle A,a\rangle$, define the value of $\Xi^H(\delta)$ at this S-algebra as

$$\Xi^H(\delta)\langle A,a\rangle = \langle HA, Ha\circ\delta_A\rangle.$$

To see this is indeed an S-algebra we examine the commutativity of the following diagrams:

$$\begin{tikzcd}
S^2HA \arrow[r, "S\delta_A"] \arrow[dd, "\mu_{HA}"'] & SHSA \arrow[r, "SHa"] \arrow[d, "\delta_{SA}"'] & SHA \arrow[d, "\delta_A"] \\
& HS^2A \arrow[r, "HSa"] \arrow[d, "H\mu_A"'] & HSA \arrow[d, "Ha"] \\
SHA \arrow[r, "\delta_A"'] & HSA \arrow[r, "Ha"'] & HA
\end{tikzcd}
\qquad
\begin{tikzcd}
HA \arrow[r, "\eta_{HA}"] \arrow[dr, "H\eta_A"] \arrow[ddr, "id"'] & SHA \arrow[d, "\delta_A"] \\
& HSA \arrow[d, "Ha"] \\
& HA
\end{tikzcd}$$

In the diagram on the left, the big square on the left commutes by (δ-μ), the upper right one by the naturality of δ, and the one on the bottom right by the associative law for the algebra a. In the diagram on the right, the upper triangle commutes by (δ-η), and the lower one by the unit law for the algebra a.

For the arrow part of $\Xi^H(\delta)$, given an S-algebra map $f : \langle A,a\rangle \to \langle B,b\rangle$, we define $\Xi^H(\delta)f$ to be $Hf : \Xi^H(\delta)\langle A,a\rangle \to \Xi^H(\delta)\langle B,b\rangle$. It is easy to see that this satisfies the diagram for algebra maps:

$$\begin{tikzcd}
SHA \arrow[r, "\delta_A"] \arrow[d, "SHf"'] & HSA \arrow[r, "Ha"] \arrow[d, "HSf"] & HA \arrow[d, "Hf"] \\
SHB \arrow[r, "\delta_B"'] & HSB \arrow[r, "Hb"'] & HB
\end{tikzcd}$$

the left square commutes by the naturality of δ, and the right one because f is an S-algebra map. The endofunctor $\Xi^H(\delta)$ clearly satisfies the conditions to be a lifting of H, and this proves the proposition. $\square$

Just as in the previous section, we can now state the following property of natural transformations $\alpha : H \to K$:

Proposition 3.16. *Given distributive laws δ^H and δ^K over endofunctors H and K and a natural transformation $\alpha : H \to K$, the following holds: if S distributes over α with respect to δ^H and δ^K then α lifts to S-Alg from $\Xi(\delta^H)$ to $\Xi(\delta^K)$.*

Proof. Assume diagram (3.1) holds for α. Then, for each S-algebra $\langle A, a\rangle$, pasting the diagram (3.1) for the component at A together with the naturality square for the arrow a we obtain diagram (3.4) for α

$$\begin{tikzcd}
SHA \arrow[r, "\delta^H_A"] \arrow[d, "S\alpha_A"'] & HSA \arrow[r, "Ha"] \arrow[d, "\alpha_{SA}"] & HA \arrow[d, "\alpha_A"] \\
SKA \arrow[r, "\delta^K_A"'] & KSA \arrow[r, "Ka"'] & KAa
\end{tikzcd}$$

showing that α lifts to *S-Alg* from $\Xi(\delta^H)$ to $\Xi(\delta^K)$. □

This amounts to say that if a natural transformation $\alpha : H \to K$ is an arrow in $\mathsf{Dist}^{\mathsf{S}}$, it has to be an arrow in $\mathsf{Lift}_{S\text{-}Alg}$, too. We now define the functor using the correspondence estabilished in Proposition 3.15 and Proposition 3.16.

Corollary 3.17. *The mapping Ξ^H in Proposition 3.15 defines a faithful functor Ξ from Dist^S to $\mathsf{Lift}_{S\text{-}Alg}$.*

Proof. Define $\Xi(H, \delta^H) = \Xi^H(\delta^H)$ and $\Xi(\alpha) = \alpha : \Xi(\delta^H) \to \Xi(\delta^K)$ for any $\alpha : \delta^H \to \delta^K$. Verifying functoriality and faithfulness is easy. □

3.5 Proving the isomorphism

We present in this section our first theorem, which states that the correspondence shown in the previous two sections is an isomorphism, or more precisely, that the functors Θ and Ξ defined in the previous two sections are mutually invertible between the category of distributive laws and that of liftings.

$$\begin{tikzcd}
\mathsf{Lift}_{S\text{-}Alg} \arrow[r, shift left, "\Theta"] & \mathsf{Dist}^{\mathsf{S}} \arrow[l, shift left, "\Xi"]
\end{tikzcd}$$

Recall that the object part of the functor Θ is defined to send $\widehat{H}$ to $\Theta(\widehat{H}) = \widehat{\mu}^{\widehat{H}} \circ SH\eta$, just as in Proposition 3.12, while the functor Ξ sends δ^H to a functor $\Xi^H(\delta^H)$ as defined in Proposition 3.15.

In the next lemma we prove that these mappings yield a bijection between objects of $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{S\text{-}Alg}$.

Proposition 3.18. *Given a monad (S, μ, η) and an endofunctor H on $\mathcal{C}$,*

1. *for any lifting $\widehat{H}$ of H to S-Alg,*

$$\widehat{H} = \Xi(\Theta(\widehat{H}))$$

holds, or equivalently, for any S-algebra $\langle A, a\rangle$,

$$\widehat{a}^{\widehat{H}} = Ha \circ \widehat{\mu}_A^{\widehat{H}} \circ SH\eta_A. \tag{3.7}$$

2. *for any distributive law $\delta : SH \to HS$ of S over H,*

$$\delta = \Theta(\Xi(\delta))$$

holds, or equivalently,

$$\delta_A = H\mu_A \circ \delta_{SA} \circ SH\eta_A. \tag{3.8}$$

Proof. For 1., we start with the equality on objects. The value of $\Xi(\Theta(\widehat{H}))$ at an S-algebra $\langle A, a\rangle$ is

$$\Xi(\Theta(\widehat{H}))\langle A, a\rangle = \langle HA, Ha \circ \Theta(\widehat{H})_A\rangle$$

where by definition the structure map decomposes as

$$Ha \circ \Theta(\widehat{H})_A = Ha \circ \widehat{\mu}_A^{\widehat{H}} \circ SH\eta_A$$

We need to show that this equals to the structure map $\widehat{a}^{\widehat{H}}$ of $\widehat{H}\langle A, a\rangle$. To see that holds, we verify that the diagram

$$\begin{tikzcd}
 & SHSA \arrow[r, "\widehat{\mu}_A^{\widehat{H}}"] \arrow[d, "SHa"] & HSA \arrow[d, "Ha"] \\
SHA \arrow[ur, "SH\eta_A"] \arrow[r, "id_{HA}"'] & SHA \arrow[r, "\widehat{a}^{\widehat{H}}"'] & HA
\end{tikzcd}$$

commutes, where the triangle on the left commutes by the unit law for an S-algebra a and the square on the right commutes by the diagram 3.5. Hence the object part of $\Xi(\Theta(\widehat{H}))$ is equal to that of $\widehat{H}$. Since the forgetful functor from S-Alg to $\mathcal{C}$ is faithful and both $\widehat{H}$ and $\Xi(\Theta(\widehat{H}))$ lifts H, equality on arrows is immediate.

Next, for 2., the component of $\Theta(\Xi(\delta^H))$ at A is given as

$$\Theta(\Xi(\delta^H))_A = \widehat{\mu}^{\Xi(\delta^H)} \circ SH\eta_A = H\mu_A \circ \delta^H_{SA} \circ SH\eta_A$$

which is shown to be equal to δ_A in the following diagram:

$$\begin{tikzcd}
SHSA \arrow[r, "\delta^H_{SA}"] & HS^2A \arrow[r, "H\mu_A"] & HSA \\
SHA \arrow[u, "SH\eta_A"] \arrow[r, "\delta^H_A"'] & HSA \arrow[u, "HS\eta_A"] \arrow[ur, "\mathrm{id}_{HSA}"'] &
\end{tikzcd}$$

The square on the left commutes by the naturality of δ^H, and the triangle on the right by the unit law of the monad. This proves $\Theta(\Xi(\delta^H)) = \delta^H$. This completes the proof of the proposition. □

Our goal in this section is the following theorem:

Theorem 3.19. *The categories* Dist^S *and* $\mathsf{Lift}_{S\text{-}Alg}$ *are isomorphic.*

Proof. We show that the functors Θ and Ξ define an isomorphism of categories between Dist^S and $\mathsf{Lift}_{S\text{-}Alg}$. Proposition 3.18 means that both Θ and Ξ are mutually inverse on objects. Since by definition they are faithful functors, they are isomorphisms of categories between Dist^S and $\mathsf{Lift}_{S\text{-}Alg}$. □

In Chapter 5 we prove the pseudo-version of this theorem, but there we do not have an isomorphism but only an equivalence of 2-categories.

3.6 Lifting of H^2

In this section we investigate the properties related to the composite endofunctor H^2, which we will need in the discussion in the next section when we consider lifting the multiplication μ of a monad. Given a lifting $\widehat{H}$ of H, it is easy to see that $\widehat{H}^2$ is a lifting of H^2. The equation (3.7) tells us that the structure map of the value, $\widehat{H}^2\langle A,a\rangle = \langle H^2A, \widehat{a}^{\widehat{H}^2}\rangle$, of $\widehat{H}^2$ at $\langle A,a\rangle$ is given by

$$\widehat{a}^{\widehat{H}^2} = H\widehat{a}^{\widehat{H}} \circ \widehat{\mu}^{\widehat{H}}_{HA} \circ SH\eta_{HA}.$$

Applying (3.7) again, this is further decomposed as

$$\widehat{a}^{\widehat{H}^2} = H^2a \circ H\widehat{\mu}^{\widehat{H}}_{A} \circ HSH\eta_A \circ \widehat{\mu}^{\widehat{H}}_{HA} \circ SH\eta_{HA}$$

and since we know $\Theta(\widehat{H}) = \mu^{\widehat{H}} \circ SH\eta$, we have

$$\widehat{a}^{\widehat{H}^2} = H^2a \circ H\Theta(\widehat{H})_A \circ \Theta(\widehat{H})_{HA}.$$

One would naturally expect the last two components of the composed arrow on the right hand side to be a distributive law. Indeed this is the case, as we see in the next lemma.

Lemma 3.20. *Given a distributive law $\delta : SH \to HS$ of a monad S and an endofunctor H on a category $\mathcal{C}$, the natural transformation*

$$SHH \xrightarrow{\delta H} HSH \xrightarrow{H\delta} HHS$$

is a distributive law of S over H^2

Proof. We only need to verify that the two conditions (δ-μ) and (δ-η) of distributive laws hold for the natural transformation $H\delta \circ \delta H : SH^2 \to H^2S$. For the associative law (δ-μ),

$$\begin{tikzcd}
S^2H^2 \arrow[rr, "S\delta H"] \arrow[dd, "\mu H^2"'] & & SHSH \arrow[rr, "SH\delta"] \arrow[drr, "\delta SH"'] & & SH^2S \arrow[rr, "\delta HS"] \arrow[d, phantom, "(1)"] & & HSHS \arrow[rr, "H\delta S"] & & H^2S^2 \arrow[dd, "H^2\mu"] \\
& & (2) & & HS^2H \arrow[urr, "HS\delta"'] \arrow[d, "H\mu H"] & & (3) & & \\
SH^2 \arrow[rrrr, "\delta H"'] & & & & HSH \arrow[rrrr, "H\delta"'] & & & & H^2S
\end{tikzcd}$$

where the area (1) commutes by the naturality of δ and (2) and (3) commute by (δ-μ) for δ.

And for the unit law (δ-η),

$$\begin{tikzcd}
H^2 \arrow[d, "\eta H^2"'] \arrow[dr, "H\eta H"] \arrow[drr, "H^2\eta"] & & \\
SH^2 \arrow[r, "\delta H"'] & HSH \arrow[r, "H\delta"'] & H^2S
\end{tikzcd}$$

where the both triangles commute by (δ-η) for δ. □

The above discussion amounts to the following proposition.

Proposition 3.21. *Given an endofunctor H on $\mathcal{C}$, a lifting $\widehat{H}$ of H to S-Alg and a distributive law δ of S over H,*

$$\widehat{H}^2 = \Xi(H\Theta(\widehat{H}) \circ \Theta(\widehat{H})H) \tag{3.9}$$

$$H\delta \circ \delta H = \Theta(\Xi(\delta)^2) \tag{3.10}$$

hold.

The above proposition leads to a further discussion: the operations appearing on the left hand side extend to strict monoidal structures on $\mathsf{Lift}_{S\text{-}Alg}$ and $\mathsf{Dist}^{\mathsf{S}}$, and the isomorphisms Θ and Ξ are both strict monoidal functors, i.e., they preserve those strict monoidal structures. In Chapter 6, we state this more formally and further investigate the pseudo-case.

3.7 Distributive laws of *S* over monads

Now we turn our attention from endofunctors H to monads T on $\mathcal{C}$ and establish a version of the theorem proved in Section 3.5 for distributive laws of S over a monad T. First we define the notion of distributive laws of S over monads and the category $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$, consisting of distributive laws over monads and monad morphisms that S distributes over. Next, we define liftings of monads to S-*Alg* and the category $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$, consisting of liftings of monads and monad morphisms that lift to S-*Alg*. Then, we prove that the functors Θ and Ξ induce an isomorphism between those categories. In the proof of the theorem, the result stated in the previous section plays an essential role.

We start with the definition of distibutive laws of S over a monad. Distributive laws over monads are those over endofunctors with two additional conditions which follow from the compatibility with the multiplication and the unit of monads. Let (S,μ^S,η^S) and (T,μ^T,η^T) be monads on $\mathcal{C}$. A distributive law of S over T is defined as follows:

Definition 3.22. Let (S,μ^S,η^S) and (T,μ^T,η^T) be monads on $\mathcal{C}$. A *distributive law* δ of S over T is a natural transformation $\delta : ST \to TS$ that makes the following diagrams commute:

$$\begin{tikzcd} S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"'] & STS \arrow[r, "\delta S"] & TS^2 \arrow[d, "T\mu^S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} T \arrow[d, "\eta^S T"'] \arrow[dr, "T\eta^S"] & \\ ST \arrow[r, "\delta"'] & TS \end{tikzcd}$$

$$\begin{tikzcd} ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"'] & STS \arrow[r, "T\delta"] & T^2S \arrow[d, "\mu^T S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} S \arrow[d, "S\eta^T"'] \arrow[dr, "\eta^T S"] & \\ ST \arrow[r, "\delta"'] & TS \end{tikzcd} \tag{3.11}$$

Just as in the case of endofunctors, distributive laws over monads form a category, in which we consider only the arrows that are monad morphisms. Recall the definition of a monad morphism defined in Section 2.1.

Proposition 3.23. *The following data form a category we denote by Dist^S_{monads}: the objects are pairs $((T,\mu^T,\eta^T),\delta)$ of a monad (T,μ^T,η^T) and a distributive law δ of S over it, and an arrow from $((T,\mu^T,\eta^T),\delta^T)$ to $((T',\mu^{T'},\eta^{T'}),\delta^{T'})$ is given by a monad morphism $\alpha : (T,\mu^T,\eta^T) \to (T',\mu^{T'},\eta^{T'})$ that S distributes over with respect to δ^T and $\delta^{T'}$.*

3.7.1 Lifting a monad to *S-Alg*

Now we define the lifting of monads: the main point is the liftings of the natural transformations μ and η.

Definition 3.24. A *lifting* of (T, μ^T, η^T) to *S-Alg* is a monad $(\widehat{T}, \mu^{\widehat{T}}, \eta^{\widehat{T}})$ for which

$$U\widehat{T} = TU \tag{3.12a}$$
$$U\mu^{\widehat{T}} = \mu^T U, \quad U\eta^{\widehat{T}} = \eta^T U \tag{3.12b}$$

hold, where U is the forgetful functor from *S-Alg* to $\mathcal{C}$.

The condition (3.12a) means that $\widehat{T}$ is a lifting of T as an endofunctor, hence, just as in the Definition 3.5, for any S-algebra $\langle A, a\rangle$ and any map of S-algebras $f : \langle A, a\rangle \to \langle B, b\rangle$, the value $\widehat{T}\langle A, a\rangle = \langle TA, \widehat{a}\rangle$ is an S-algebra and the structure map $\widehat{a} : STA \to TA$ satisfies the diagrams (3.3a) for $H = T$. And $Tf : \widehat{T}\langle A, a\rangle \to \widehat{T}\langle B, b\rangle$ is an S-algebra map, satisfying the diagram (3.3b) for $H = T$.

Since T is a monad, in addition, we have conditions (3.12b), which tell us that $\widehat{T}$ must be a lifting of T as an endofunctor such that the components of $\mu^{\widehat{T}}$ and $\eta^{\widehat{T}}$ at an S-algebra $\langle A, a\rangle$, which are S-algebra maps,

$$\begin{aligned} \mu^{\widehat{T}}_{\langle A,a\rangle} &: \widehat{T}^2\langle A, a\rangle \longrightarrow \widehat{T}\langle A, a\rangle \\ \eta^{\widehat{T}}_{\langle A,a\rangle} &: Id_{S\text{-}Alg} \longrightarrow \widehat{T}\langle A, a\rangle \end{aligned} \tag{3.13}$$

are given by the components μ^T_A and η^T_A of μ^T and η^T, respectively. Since, $\widehat{T}^2$ is a lifting of T^2, and $Id_{S\text{-}Alg}$ is a lifting of $Id_{\mathcal{C}}$, the condition (3.13) means that $\mu^{\widehat{T}}$ and $\eta^{\widehat{T}}$ are liftings of μ^T and η^T to *S-Alg*, in the sense of Definition 3.7, with respect to these liftings.

Here, again, we only consider the monad morphisms as arrows. The following property of liftings of natural transformations defined in Definition 3.7 is easily verified.

Lemma 3.25. *Given a monad morphism* $\alpha : (T, \mu^T, \eta^T) \to (T', \mu^{T'}, \eta^{T'})$*, if the natural transformation* $\alpha : T \to T'$ *lifts to S-Alg from* $\widehat{T}$ *to* $\widehat{T}'$ *as endofunctors, then it is a monad morphism from* $(\widehat{T}, \mu^{\widehat{T}}, \eta^{\widehat{T}})$ *to* $(\widehat{T}', \mu^{\widehat{T}'}, \eta^{\widehat{T}'})$.

Now we consider a category of liftings of monads and monad morphisms between them:

Proposition 3.26. *The following data form a category we denote by* $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$*: objects are pairs* $((T, \mu^T, \eta^T), \widehat{T})$ *of a monad* (T, μ^T, η^T) *and its lifting* $\widehat{T}$ *as a monad to S-Alg,* *and an arrow from* $((T,\mu^T,\eta^T),\widehat{T})$ *to* $((T',\mu^{T'},\eta^{T'}),\widehat{T}')$ *is given by a monad morphism* $\alpha : (T,\mu^T,\eta^T) \to (T',\mu^{T'},\eta^{T'})$ *that lifts to S-Alg from* $\widehat{T}$ *to* $\widehat{T}'$.

3.7.2 Isomorphism for the monad case

Evidently, a monad is an endofunctor with additional structure. Both the distributive laws over monads and the liftings of monads are defined accordingly. In Section 3.5 we saw that the functors Θ and Ξ define an isomorphism of categories between $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{\textit{S-Alg}}$. In this section, we consider the categories $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$ and $\mathsf{Lift}^{\mathsf{monads}}_{\textit{S-Alg}}$, and show that the functors Θ and Ξ naturally induce functors that again define an isomorphism between those categories.

Proposition 3.27. *The functor* Θ *in Proposition 3.12 induces a functor from* $\mathsf{Lift}^{\mathsf{monads}}_{\textit{S-Alg}}$ *to* $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$.

Proof. We define

$$\Theta^{\mathsf{m}} : \mathsf{Lift}^{\mathsf{monads}}_{\textit{S-Alg}} \to \mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$$

as $\Theta^{\mathsf{m}}((T,\mu^T,\eta^T),\widehat{T}) = ((T,\mu^T,\eta^T),\Theta(\widehat{T}))$ and $\Theta^{\mathsf{m}}(\alpha) = \alpha$. It is obvious that an arrow α in $\mathsf{Lift}^{\mathsf{monads}}_{\textit{S-Alg}}$ is necessarily an arrow in $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$. Then what we are left to show is that, the distributive law $\Theta(\widehat{T})$ satisfies the extra conditions (3.11) for the distributivity over a monad. In order to do so, we only need to note that, by the definition of liftings of monads, both $\mu^T : T^2 \to T$ and $\eta^T : Id \to T$ lift to *S-Alg* (3.13). This implies both μ^T and η^T are arrows in $\mathsf{Lift}_{\textit{S-Alg}}$. Hence $\Theta(\mu^T) = \mu^T$ is an arrow in $\mathsf{Dist}^{\mathsf{S}}$ such that, for any object A of $\mathcal{C}$,

$$\begin{tikzcd}
ST^2A \arrow[r, "\Theta(\widehat{T}^2)_A"] \arrow[d, "S\mu^T_A"'] & T^2SA \arrow[d, "\mu^T_{SA}"] \\
STA \arrow[r, "\Theta(\widehat{T})_A"'] & TSA
\end{tikzcd}$$

commutes. Since $\Theta(\widehat{T}^2) = T\Theta(\widehat{T}) \circ \Theta(\widehat{T})T$ follows from Proposition 3.21, we have the diagram

$$\begin{tikzcd} ST^2A \arrow[r, "\Theta(\widehat{T})_{TA}"] \arrow[d, "S\mu^T_A"'] & TSTA \arrow[r, "T(\Theta(\widehat{T}))_A"] & T^2SA \arrow[d, "\mu^T_{SA}"] \\ STA \arrow[rr, "\Theta(\widehat{T})_A"'] & & TSA \end{tikzcd}$$

commute, which is indeed the associative axiom for $\Theta(\widehat{T})$ and μ^T. A similar argument holds for η^T and the diagram

$$\begin{tikzcd} SA \arrow[r, "id_{SA}"] \arrow[d, "S\eta^T_A"'] & SA \arrow[d, "\eta^T_{SA}"] \\ STA \arrow[r, "\Theta(\widehat{T})_A"'] & TSA \end{tikzcd}$$

holds. Note $\Theta(Id_{S\text{-}Alg}) = Id_S$. Hence $\Theta(\widehat{T})$ is indeed a distributive law over a monad. This proves that Θ^{m} is a functor from $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$ to $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$. $\square$

Proposition 3.28. *The functor Ξ in Proposition 3.12 induces a functor from $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$ to $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$.*

Proof. from $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$ to $\mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}}$. We define a functor

$$\Xi^{\mathsf{m}} : \mathsf{Dist}^{\mathsf{S}}_{\mathsf{monads}} \to \mathsf{Lift}^{\mathsf{monads}}_{S\text{-}Alg}$$

as $\Xi^{\mathsf{m}}((T,\mu^T,\eta^T),\delta^T) = ((T,\mu^T,\eta^T),(\Xi(\delta^T),\mu^{\Xi(\delta^T)},\eta^{\Xi(\delta^T)}))$ and $\Xi^{\mathsf{m}}(\alpha) = \alpha$. We need to show that μ^T and η^T lift to S-Alg from $\Xi(\delta^T)^2$ to $\Xi(\delta^T)$, and, from $\Xi(Id_{\mathcal{C}})$ to $\Xi(\delta^T)$, respectively. We verify that for both cases the suitable instances of the diagram (3.4) commute. For the multiplication μ^T, it should lift from $\Xi(\delta^T)^2$ to $\Xi(\delta^T)$. From Proposition 3.21 it follows that $\Xi(\delta^T)^2 = \Xi(T\delta^T \circ \delta^T T)$ holds, so for any S-algebra $\langle A,a\rangle$, the equality $\widehat{a}^{\Xi(\delta^T)^2} = \widehat{a}^{\Xi(T\delta^T\circ\delta^T T)}$ holds. Therefore, the diagram (3.4) for μ^T is given as

$$\begin{tikzcd} ST^2A \arrow[r, "\delta^T_{TA}"] \arrow[d, "S\mu^T_A"'] & TSTA \arrow[r, "T\delta^T_A"] & T^2SA \arrow[r, "T^2a"] \arrow[d, "\mu^T_{TA}"'] & T^2A \arrow[d, "\mu^T_A"] \\ STA \arrow[rr, "\delta^T_A"'] & & TSA \arrow[r, "Ta"'] & TA \end{tikzcd}$$

which commutes, by the associative law δ^T-μ^T for distributive laws and by the naturality of μ^T. For the unit η^T, since $\Xi(Id_{\mathcal{C}}) = Id_{S\text{-}Alg}$ holds, the diagram is given as

$$\begin{tikzcd}
SA \arrow[r, "id_{SA}"] \arrow[d, "S\eta^T_A"'] & SA \arrow[r, "a"] \arrow[d, "\eta^T_{SA}"] & A \arrow[d, "\eta^T_A"] \\
STA \arrow[r, "\delta^T_A"'] & TSA \arrow[r, "Ta"'] & TA
\end{tikzcd}$$

which commutes by the unit law δ^T-η^T and the naturality of η^T. For the arrow part of Ξ^{m}, again, it is easily seen to be well-defined. □

From the above two propositions and Theorem 3.19, we have the following result.

Corollary 3.29. *The categories $\mathsf{Dist}^S_{\mathsf{monads}}$ and $\mathsf{Lift}^{\mathsf{monads}}_{S\text{-}\mathsf{Alg}}$ are isomorphic.*

3.8 The composite monad *TS*

The composite TS of two monads (S, μ^S, η^S) and (T, μ^T, η^T) on a category $\mathcal{C}$ is not necessarily a monad, but when there exists a distributive law of S over T, it is the case that TS is a monad on $\mathcal{C}$. In this section we prove this fact, and then investigate the relation between the algebras of this monad TS and the algebras of the lifting of T induced by the distributive law.

Proposition 3.30. *Given a distributive law $\delta : ST \to TS$ of a monad (S, μ^S, η^S) over a monad (T, μ^T, η^T) on a ordinary category $\mathcal{C}$, the composite functor TS acquires the structure for a monad on $\mathcal{C}$, with multiplication given by*

$$TSTS \xrightarrow{T\delta S} TTSS \xrightarrow{\mu^T \mu^S} TS$$

Proof. We define the multiplication μ^{TS} as above and the unit η^{TS} as $\eta^{TS} : Id \xrightarrow{\eta^T \eta^S} TS$ and claim that $(TS, \mu^{TS}, \eta^{TS})$ is a monad on $\mathcal{C}$. Obviously, by definition,

$$\mu^T \mu^S = T\mu^S \circ \mu^T S^2 = \mu^T S \circ T^2 \mu^S$$

holds. It does not matter which to choose in the actual calculation, but in the following we agree that we always use the second decomposition, i.e., we always calculate S first.

Similarly for $\eta^T\eta^S$. Then, we are left to prove that μ^{TS} and η^{TS} satisfy the axioms for monads. For the associative law, the diagram

$$
\begin{tikzcd}
TSTSTS \arrow[r, "TST\delta S"] \arrow[d, "T\delta STS"'] \arrow[dr, phantom, "(1)"] & TST^2S^2 \arrow[r, "TST^2\mu^S"] \arrow[d, "T\delta TS^2"] \arrow[dr, phantom, "(4)"] & TST^2S \arrow[r, "TS\mu^T S"] \arrow[d, "T\delta TS"] & TSTS \arrow[dd, "T\delta S"] \\
T^2S^2TS \arrow[r, "T^2S\delta S"'] \arrow[dd, "T^2\mu^S TS"'] & T^2STS^2 \arrow[r, "T^2ST\mu^S"'] \arrow[d, "T^2\delta S^2"] \arrow[dr, phantom, "(5)"] & T^2STS \arrow[d, "T^2\delta S"] \arrow[r, phantom, "(8)"] & {} \\
{} \arrow[r, phantom, "(2)"] & T^3S^3 \arrow[r, "T^3S\mu^S"'] \arrow[d, "T^3\mu^S S"] \arrow[dr, phantom, "(6)"] & T^3S^2 \arrow[r, "T\mu^T S^2"'] \arrow[d, "T^3\mu^S"] \arrow[dr, phantom, "(9)"] & T^2S^2 \arrow[d, "T^2\mu^S"] \\
T^2STS \arrow[r, "T^2\delta S"'] \arrow[d, "\mu^T STS"'] \arrow[dr, phantom, "(3)"] & T^3S^2 \arrow[r, "T^3\mu^S"'] \arrow[d, "\mu^T TS^2"] \arrow[dr, phantom, "(7)"] & T^3S \arrow[r, "T\mu^T S"'] \arrow[d, "\mu^T TS"] \arrow[dr, phantom, "(10)"] & T^2S \arrow[d, "\mu^T S"] \\
TSTS \arrow[r, "T\delta S"'] & T^2S^2 \arrow[r, "T^2\mu^S"'] & T^2S \arrow[r, "\mu^T S"'] & TS
\end{tikzcd}
$$

commutes for the following reasons: the squares (1), (4) and (5) by the naturality of δ, (3), (7) and (9) by the naturality of μ^T, (2) and (8) by the axioms for distributive law δ-μ^S and δ-μ^T, respectively, and (6) and (10) by the associative laws for monads S and T, respectively. Next, the left unit law is given as

$$
\begin{tikzcd}
TS \arrow[r, "TS\eta^S"] \arrow[ddr, "id_{TS}"'] \arrow[dr, phantom, "(1)"] & TS^2 \arrow[r, "TS\eta^T S"] \arrow[dd, "T\mu^S"] \arrow[dr, "T\eta^T S^2"] & TSTS \arrow[d, "T\delta S"] \\
{} & {} \arrow[r, phantom, "(3)"] & T^2S^2 \arrow[d, "T^2\mu^S"] \\
{} & TS \arrow[r, "T\eta^T S"] \arrow[dr, "id_{TS}"'] & T^2S \arrow[d, "\mu^T S"] \\
{} & {} & TS
\end{tikzcd}
$$

which commutes (1) and (4) by the left unit laws for monads S and T, respectively, and (2) by the axiom for distributive law δ-η^T and (3) by the naturality of η^T. Finally, the right unit law:

$$\begin{tikzcd}
TSTS \arrow[d, "T\delta S"'] \arrow[dr, phantom, "(5)"] & STS \arrow[l, "\eta^T STS"'] \arrow[d, "\delta S"] \arrow[r, phantom, "(8)"] & TS \arrow[l, "\eta^S TS"'] \arrow[dl, "T\eta^S S"'] \arrow[ddl, "\mathrm{id}_{TS}"] \\
T^2S^2 \arrow[d, "T^2\mu^S"'] \arrow[dr, phantom, "(6)"] & TS^2 \arrow[l, "\eta^T TS^2"'] \arrow[d, "T\mu^S"] \arrow[r, phantom, "(9)"] & \\
T^2S \arrow[d, "\mu^T S"'] \arrow[dr, phantom, "(7)"] & TS \arrow[l, "\eta^T TS"'] \arrow[dl, "\mathrm{id}_{TS}"] & \\
TS & &
\end{tikzcd}$$

the commutativity is given by: (5) and (6) by the naturality of η^T, (9) and (7) by the right unit laws for monads S and T, respectively, and (8) by the axiom for distributive law δ-η^T. This concludes the proof that $(TS, \mu^{TS}, \eta^{TS})$ is a monad. $\square$

3.8.1 Comparison between the algebras

Having seen that TS is a monad under the exisitence of a distributive law δ of S over T, now we consider its algebras and the relationship between the category of TS-algebras and that of the algebras of the lifting induced by the distributive law.

First we have a look at the algebra of the composite monad TS. A TS-algebra $\langle A, l : TSA \to A\rangle$ is, by definition, a pair of an object A of $\mathcal{C}$ and an arrow $l : TSA \to A$ such that the following diagrams commute:

$$\begin{tikzcd}
TSTSA \arrow[r, "T\delta_{SA}"] \arrow[d, "TSl"'] & T^2S^2A \arrow[r, "T^2\mu^S_A"] & T^2SA \arrow[r, "\mu^T_{SA}"] & TSA \arrow[d, "l"] \\
TSA \arrow[rrr, "l"'] & & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta^S_A"] \arrow[drr, "id_A"'] & SA \arrow[r, "\eta^T_{SA}"] & TSA \arrow[d, "l"] \\
& & A
\end{tikzcd} \tag{3.14}$$

And a map $f : \langle A, l\rangle \to \langle A', l'\rangle$ of TS-algebras is an arrow $f : A \to A'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} TSA \arrow[r, "TSf"] \arrow[d, "l"'] & TSA' \arrow[d, "l"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

commutes. These define the category *TS-Alg*.

Now, in turn, we consider the algebras of a monad on the category *S-Alg* which is given as the lifting $\Xi(\delta)$ of T induced by the distributive law δ. A $\Xi(\delta)$-algebra has the form of $\langle\langle A,h\rangle,k\rangle$, where $\langle A,h : SA \to A\rangle$ is an S-algebra and the structure map k is an S-algebra map $k : \langle TA, Th \circ \delta_A\rangle \to \langle A,h\rangle$, which means that k is an arrow $k : TA \to A$ in $\mathcal{C}$ such that the following diagram should commute:

$$\begin{tikzcd} STA \arrow[r, "\delta_A"] \arrow[d, "Sk"'] & TSA \arrow[r, "Th"] & TA \arrow[d, "k"] \\ SA \arrow[rr, "h"'] & & A \end{tikzcd}$$

Moreover, the arrow k should also satisfy the commutativity of the following two diagrams, in order to be a structure map of a $\Xi(\delta)$-algebra:

$$\begin{tikzcd} \Xi(\delta)^2\langle A,h\rangle \arrow[r, "\Xi(\delta)k"] \arrow[d, "\mu^{\Xi(\delta)}_{\langle A,h\rangle}"'] & \Xi(\delta)\langle A,h\rangle \arrow[d, "k"] \\ \Xi(\delta)\langle A,h\rangle \arrow[r, "k"'] & \langle A,h\rangle \end{tikzcd} \qquad \begin{tikzcd} \langle A,h\rangle \arrow[r, "\eta^{\Xi(\delta)}_{\langle A,h\rangle}"] \arrow[dr, "id_{\langle A,h\rangle}"'] & \Xi(\delta)\langle A,h\rangle \arrow[d, "k"] \\ & \langle A,h\rangle \end{tikzcd}$$

Since all the arrows in the above diagram represent S-algebra maps, the requirements reduce to the commutativity of the following two diagrams:

$$\begin{tikzcd} T^2A \arrow[r, "Tk"] \arrow[d, "\mu^T_A"'] & TA \arrow[d, "k"] \\ TA \arrow[r, "k"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta^T_A"] \arrow[dr, "id_A"'] & TA \arrow[d, "k"] \\ & A \end{tikzcd}$$

which means that the arrow k has the property of a structure map of T-algebras, i.e., $\langle A,k\rangle$ forms a T-algebra. And a map $f : \langle\langle A,h\rangle,k\rangle \to \langle\langle A',h'\rangle,k'\rangle$ of $\Xi(\delta)$-algebras is an S-algebra map $f : \langle A,h\rangle \to \langle A',h'\rangle$ which makes the diagram

$$\begin{tikzcd}
\Xi(\delta)\langle A,h\rangle \arrow[r, "\Xi(\delta) f"] \arrow[d, "k"'] & \Xi(\delta)\langle A',h'\rangle \arrow[d, "k'"] \\
\langle A,h\rangle \arrow[r, "f"'] & \langle A',h'\rangle
\end{tikzcd}$$

commute. All the arrows in this diagram are S-algebra maps, however the commutativity of this square only depends on the commutativity of the square

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "k"'] & TA' \arrow[d, "k'"] \\
A \arrow[r, "f"'] & A'
\end{tikzcd}$$

which means that $f : A \to A'$ is also a T-algebra map, not only an S-algebra map.

The opposite direction of the above discussion also holds; to summarise, we have the following proposition:

Proposition 3.31. *Given an object A and arrows $h : SA \to A$ and $k : TA \to A$ of $\mathcal{C}$, a structure of the form $\langle\langle A,h\rangle,k\rangle$ is a $\Xi(\delta)$-algebra if and only if*

1. *$\langle A,h\rangle$ is an S-algebra*
2. *$k : TA \to A$ is an S-algebra map from $\langle TA, Th\circ\delta_A\rangle$ to $\langle A,h\rangle$*
3. *$\langle A,k\rangle$ is a T-algebra,*

and, given an arrow $g : A \to A'$ and $\Xi(\delta)$-algebras $\langle\langle A,h\rangle,k\rangle$ to $\langle\langle A',h'\rangle,k'\rangle$, the arrow g is a $\Xi(\delta)$-algebra map from $\langle\langle A,h\rangle,k\rangle$ to $\langle\langle A',h'\rangle,k'\rangle$ if and only if

1. *g is an S-algebra map from $\langle A,h\rangle$ to $\langle A',h'\rangle$*
2. *g is a T-algebra map from $\langle A,k\rangle$ to $\langle A',k'\rangle$.*

All the $\Xi(\delta)$-algebras and maps between them again form a category which we denote by $\Xi(\delta)$-*Alg*.

Now, in the rest of the section we investigate the relationship between these two categories of algebras and establish that there exists a canonical isomorphism between them. We start with considering the adjunctions that the monads involved define.

Proposition 3.32. *There exists a unique comparison functor K from $\Xi(\delta)$-Alg to TS-Alg.*

Proof. From Lemma 2.4, for the monad (S, μ^S, η^S) on $\mathcal{C}$ and $(\Xi(\delta), \mu^{\Xi(\delta)}, \eta^{\Xi(\delta)})$ on S-*Alg*, we have adjunctions

$$(F^S, G^S, \eta^S, \varepsilon^S) : \mathcal{C} \underset{F^S}{\overset{G^S}{\leftrightarrows}} S\text{-}Alg$$

$$(F^{\Xi(\delta)}, G^{\Xi(\delta)}, \eta^{\Xi(\delta)}, \varepsilon^{\Xi(\delta)}) : S\text{-}Alg \underset{F^{\Xi(\delta)}}{\overset{G^{\Xi(\delta)}}{\leftrightarrows}} \Xi(\delta)\text{-}Alg.$$

Then, Lemma 2.6 says that these two adjunctions yield a composite adjunction

$$(F^{\Xi(\delta)S}, G^{\Xi(\delta)S}, \eta^{\Xi(\delta)S}, \varepsilon^{\Xi(\delta)S}) : \mathcal{C} \underset{F^{\Xi(\delta)}F^S}{\overset{G^S G^{\Xi(\delta)}}{\leftrightarrows}} \Xi(\delta)\text{-}Alg$$

as defined in the lemma. Then we can in turn use Lemma 2.5 to construct a monad $G^{\Xi(\delta)S}F^{\Xi(\delta)S}$ from this adjunction. Now it is routine to show that this monad in fact equals the composite monad $(TS, \mu^{TS}, \eta^{TS})$. Therefore, we can use Lemma 2.7 to obtain the comparison functor K, which is required in the proposition. $\square$

Therefore, we have a unique comparison functor

$$K : \Xi(\delta)\text{-}Alg \longrightarrow TS\text{-}Alg$$

such that $KF^{\Xi(\delta)S} = F^{TS}$ and $G^{TS}K = G^{\Xi(\delta)S}$, between the above adjunction

$$(F^{\Xi(\delta)S}, G^{\Xi(\delta)S}, \eta^{\Xi(\delta)S}, \varepsilon^{\Xi(\delta)S})$$

and the adjunction $(F^{TS}, G^{TS}, \eta^{TS}, \varepsilon^{TS})$, which is induced by the monad $(TS, \mu^{TS}, \eta^{TS})$. Lemma 2.7 designates the construction of K as follows: for a $\Xi(\delta)$-algebra $\langle\langle A, h\rangle, k\rangle$,

$$K\langle\langle A, h\rangle, k\rangle = \langle G^S G^{\Xi(\delta)}\langle\langle A, h\rangle, k\rangle, G^S G^{\Xi(\delta)}(\varepsilon^{\Xi(\delta)} \cdot F^{\Xi(\delta)}\varepsilon^S G^{\Xi(\delta)})_{\langle\langle A,h\rangle,k\rangle}\rangle$$

which reduces to $\langle A, k \circ Th\rangle$. And a $\Xi(\delta)$-algebra map $f : \langle\langle A, h\rangle, k\rangle \to \langle\langle A', h'\rangle, k'\rangle$, where $f : A \to A'$ is both an S-algebra map and a T-algebra map, is sent to a TS-algebra map

$$Kf : \langle A, k \circ Th\rangle \to \langle A', k' \circ Th'\rangle.$$

What we next need to show is that this comparison functor K is an isomorphism. To this end, we now define an inverse K^{-1} of K,

$$K^{-1} : TS\text{-}Alg \to \Xi(\delta)\text{-}Alg.$$

Given $\langle A, l : TSA \to A\rangle$ a TS-algebra, we define the object part of K^{-1} as;

$$K\langle A, l\rangle = \langle\langle A, l \circ \eta^T_{SA}\rangle, l \circ T\eta^S_A\rangle.$$

The following lemma verifies the well-definedness of the above definition.

Lemma 3.33. *The algebra* $\langle\langle A, l \circ \eta^T_{SA}\rangle,\ l \circ T\eta^S_A\rangle$ *is a* $\Xi(\delta)$*-algebra.*

Proof. We prove the statement in three steps.
1. We first show that $\langle A, l \circ \eta^T_{SA}\rangle$ is an S-algebra. The associative law for the algebra is given in the left diagram below, and the unit law in the right diagram.

$$\begin{tikzcd}
S^2A \arrow[r, "S\eta^T_{SA}"] \arrow[dd, "\mu^S"'] \arrow[ddr, phantom, "(1)"] & STSA \arrow[r, "Sl"] \arrow[d, "\eta^T_{STSA}"'] \arrow[dr, phantom, "(2)"] & SA \arrow[d, "\eta^T_{SA}"] \\
 & TSTSA \arrow[r, "TSl"] \arrow[d, "\mu^{TS}"'] \arrow[dr, phantom, "(3)"] & TSA \arrow[d, "l"] \\
SA \arrow[r, "\eta^T_{SA}"'] & TSA \arrow[r, "l"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta^S_A"] \arrow[ddr, "id_A"'] & SA \arrow[d, "\eta^T_{SA}"] \\
 & TSA \arrow[d, "l"] \\
 & A
\end{tikzcd}$$

In the square for the associativity, the commutativity of the square (1) in the left is demonstrated in the next diagram. The right top square (2) commutes by the naturality of η^T, and the right bottom square (3) commutes by the diagram on the left in (3.14), the associative law for TS-algebras. For the diagram on the right, note that $\eta^T S \circ \eta^S$ is η^{TS}; the diagram on the right above commutes immediately by the diagram on the right in (3.14), the unit law for the structure map l.

The commutativity of the square (1) above is proved as follows:

$$
\begin{tikzcd}
S^2A \arrow[rr, "S\eta^T_{SA}"] \arrow[dr, "\eta^T_{S^2A}"'] \arrow[dddd, "\mu^S_A"'] & & STSA \arrow[dl, "\delta_{SA}"'] \arrow[d, "\eta^T_{STSA}"] \\
& TS^2A \arrow[r, phantom, "(b)"] \arrow[u, phantom, "(a)"] \arrow[dd, "T\mu^S_A"'] \arrow[dr, "\eta^T_{TS^2A}"'] & TSTSA \arrow[d, "T\delta_{SA}"] \\
\arrow[r, phantom, "(c)"] & \arrow[dr, phantom, "(d)"] & T^2S^2A \arrow[d, "T^2\mu^S_A"] \\
& TSA \arrow[r, "\eta^T_{TSA}"] \arrow[dr, "id_{TSA}"'] & T^2SA \arrow[d, "\mu^T_{SA}"] \arrow[dl, phantom, "(e)"] \\
SA \arrow[rr, "\eta^T_{SA}"'] & & TSA
\end{tikzcd}
$$

The reason of the commutativity of each area is given as follows: (a) by the unit law for the distributive law δ and η^T, (b), (c) and (d) by the naturality of η^T, and (e) by the unit law for the monad T.

2. Next, we show that $l \circ T\eta^S_A$ is an S-algebra map. The proof for this map to be an S-algebra map is of type

$$\langle TA, T(l \circ \eta^T_{SA}) \circ \delta_A \rangle \to \langle A, l \circ \eta^T_{SA} \rangle,$$

is given by the following diagram : the commutativity of the square $(*)$, which follows from the coherence theorem of distributive law, is demonstrated in the next diagram, and the squares (1) and (3) commute by that of η^T and η^S, respectively. For the squares (2) and (4), they commute because l is a TS-algebra, i.e., by the associative law for the structure map l. Note that by definition $\mu^{TS} = \mu^T S \circ T^2\mu^S \circ \delta$.

$$
\begin{tikzcd}
STA \arrow[rr, "ST\eta^S_A"] \arrow[d, "\delta_A"'] & & STSA \arrow[r, "Sl"] \arrow[d, "\eta^T_{STSA}"'] \arrow[dr, phantom, "(1)"] & SA \arrow[d, "\eta^T_{SA}"] \\
TSA \arrow[d, "T\eta^T_{SA}"'] & (*) & TSTSA \arrow[r, "TSl"'] \arrow[d, "\mu^{TS}_A"'] \arrow[dr, phantom, "(2)"] & TSA \arrow[dd, "l"] \\
T^2SA \arrow[r, "T\eta^S_{TSA}"] \arrow[d, "Tl"'] \arrow[dr, phantom, "(3)"] & TSTSA \arrow[r, "\mu^{TS}_A"] \arrow[d, "TSl"] \arrow[dr, phantom, "(4)"] & TSA \arrow[dr, "l"] & \\
TA \arrow[r, "T\eta^S_A"'] & TSA \arrow[rr, "l"'] & & A
\end{tikzcd}
$$

And for the square labelled $(*)$, we have the following commutative diagram hold mainly by the unit laws of the monads T and S, to remove μ^T's and μ^S's, by one unit law for δ and η^S, and by several naturality squares:

$$
\begin{tikzcd}
ST \arrow[rrrr, "ST\eta^S"] \arrow[d, "\delta"'] & & & & STS \arrow[dl, "\delta S"'] \arrow[d, "\eta^T STS"] \\
TS \arrow[rrr, "TS\eta^S"] \arrow[d, "T\eta^T S"'] \arrow[drrr, "id"] \arrow[dddrrrr, "id"] & & & TS^2 \arrow[d, "T\mu^S"'] \arrow[dr, "\eta^T TS^2"] & TSTS \arrow[d, "T\delta S"] \\
T^2S \arrow[dd, "T\eta^S TS"'] \arrow[ddr, "T^2\eta^S S"] \arrow[ddrr, "id"] & & & TS \arrow[dr, "\eta^T TS"] \arrow[ddr, "id"] & T^2S^2 \arrow[d, "T^2\mu^S"] \\
& & & & T^2S \arrow[d, "\mu^T S"] \\
TSTS \arrow[r, "T\delta S"'] & T^2S^2 \arrow[r, "T^2\mu^S"'] & T^2S \arrow[rr, "\mu^T S"'] & & TS
\end{tikzcd}
$$

3. The last step is to show that $\langle A, l \circ T\eta^S_A \rangle$ is a T-algebra. For this we need to show

two things. The first, the associative law, is shown below;

$$\begin{tikzcd}
T^2A \arrow[r, "T^2\eta^S_A"] \arrow[dd, "\mu^T_A"'] \arrow[ddr, phantom, "(3)"] & T^2SA \arrow[r, "Tl"] \arrow[d, "T\eta^S_{TSA}"'] \arrow[dr, phantom, "(1)"] & TA \arrow[d, "T\eta^S_A"] \\
 & TSTSA \arrow[r, "TSl"] \arrow[d, "\mu^{TS}"'] \arrow[dr, phantom, "(2)"] & TSA \arrow[d, "l"] \\
TA \arrow[r, "T\eta^S_A"'] & TSA \arrow[r, "l"'] & A
\end{tikzcd}$$

The square (1) commutes by the naturality of η^S, (2) by the fact that l is a structure map of a TS-algebra. And for the rectangle (3), we have a closer look in the diagram below on the left: the rectangle (a) commutes by the naturality of μ^T. For the remaining two triangles, (b) is the unit triangle for the distributive law δ and η^S and (c) is the right unit law for the monad S. Secondly, the unit law for $\langle A, l \circ T\eta^S_A \rangle$ is given straightforwardly by the diagram below on the right, the upper triangle of which is the equality by the definition of η^{TS} and that of the horizontal composition of natural transformations, and the lower one by the unit law for l being TS-algebra map.

$$\begin{tikzcd}
T^2A \arrow[r, "T^2\eta^S_A"] \arrow[dd, "\mu^T_A"'] \arrow[ddr, phantom, "(a)"] & T^2SA \arrow[r, "T\eta^S_{TSA}"] \arrow[d, "T^2id_{SA}"'] \arrow[dr, "T^2\eta^S_{SA}"] & TSTSA \arrow[d, "T\delta_{SA}"] \\
 & T^2SA \arrow[d, "\mu^T_{SA}"'] & T^2S^2A \arrow[l, "T^2\mu^S_A"] \\
TA \arrow[r, "T\eta^S_A"'] & TSA &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta^T_A"] \arrow[dr, "\eta^{TS}"] \arrow[ddr, "id_A"'] & TA \arrow[d, "T\eta^S_A"] \\
 & TSA \arrow[d, "l"] \\
 & A
\end{tikzcd}$$

Now it follows from Proposition 3.31 that $\langle\langle A, l \circ \eta^T_{SA} \rangle,\ l \circ T\eta^S_A \rangle$ is a $\Xi(\delta)$-algebra. $\square$

We are yet to define the arrow part of K^{-1}. A map

$$g : \langle A, l : TSA \to A \rangle \to \langle A', l' : TSA' \to A' \rangle$$

of TS-algebras is sent by K^{-1} to g itself, which is a $\Xi(\delta)$-algebra map such that

$$K^{-1}g : \langle\langle A, l \circ \eta^T_{SA} \rangle, l \circ T\eta^S_A \rangle \to \langle\langle A', l' \circ \eta^T_{SA'} \rangle, l' \circ T\eta^T_{A'} \rangle$$

To see that this indeed is a $\Xi(\delta)$-algebra map, we only need to verify that $g : A \to A'$ is both an S-algebra map and a T-algebra map, with respect to the respective structure maps. This is verified easily, as shown below, by the naturality of η^T and η^S (the top squares (1) and (3)), and by the fact that g is a TS-algebra map (the bottom ones (2) and (4)).

$$\begin{tikzcd}
SA \arrow[r, "Sg"] \arrow[d, "\eta^T_{SA}"'] \arrow[dr, phantom, "(1)"] & SA' \arrow[d, "\eta^T_{SA'}"] & TA \arrow[r, "Tg"] \arrow[d, "T\eta^S_A"'] \arrow[dr, phantom, "(3)"] & TA' \arrow[d, "T\eta^S_{A'}"] \\
TSA \arrow[r, "TSg"] \arrow[d, "l"'] \arrow[dr, phantom, "(2)"] & TSA' \arrow[d, "l'"] & TSA \arrow[r, "TSg"] \arrow[d, "l"'] \arrow[dr, phantom, "(4)"] & TSA' \arrow[d, "l'"] \\
A \arrow[r, "g"'] & A' & A \arrow[r, "g"'] & A'
\end{tikzcd}$$

From the above discussion, it follows that the functor K^{-1} is well-defined.

Having defined the functor K^{-1}, we are now left to see that it is indeed an inverse of K.

Lemma 3.34. *The comparison functor K is invertible.*

Proof. We show that $K^{-1} \circ K = id$ and $K \circ K^{-1} = id$. We start from proving the first equation. For the object part, let $\langle\langle A, h\rangle, k\rangle$ be a $\Xi(\delta)$-algebra. Then

$$\begin{aligned}
K^{-1} \circ K(\langle\langle A, h\rangle, k\rangle) &= K^{-1}(\langle A, k \circ Th\rangle) \\
&= \langle\langle A, k \circ Th \circ \eta^T_{SA}\rangle, k \circ Th \circ T\eta^S_A\rangle
\end{aligned}$$

which is equal to $\langle\langle A, h\rangle, k\rangle$ because, first the arrow $k \circ Th \circ T\eta^S_A$ is equal to k as in the diagram

$$\begin{tikzcd}
TA \arrow[d, "T\eta^S_A"'] \arrow[dr, "Tid_A"] & & \\
TSA \arrow[r, "Th"'] & TA \arrow[r, "k"'] & A
\end{tikzcd}$$

which commutes by the unit law of the S-algebra h, and, secondly, the arrow $k \circ Th \circ \eta^T_{SA}$ is equal to h because of the naturality of η^T (the square on the left) and of the unit law of the T-algebra k (the triangle):

$$
\begin{tikzcd}
SA \arrow[r, "h"] \arrow[d, "\eta^T_{SA}"'] & A \arrow[d, "\eta^T_A"'] \arrow[dr, "id_A"] & \\
TSA \arrow[r, "Th"'] & TA \arrow[r, "k"'] & A
\end{tikzcd}
$$

It is also easy to see that the arrow part of $K^{-1} \circ K$ is an identity. Hence we have $K^{-1} \circ K = id$.

For the proof of the second equation $K \circ K^{-1} = id$, let $\langle A, l \rangle$ be a TS-algebra. Then

$$
\begin{aligned}
K \circ K^{-1}(\langle A, l \rangle) &= K(\langle \langle A, l \circ \eta^T_{SA} \rangle, l \circ T\eta^S_A \rangle) \\
&= \langle A, l \circ T\eta^S_A \circ T(l \circ \eta^T_{SA}) \rangle
\end{aligned}
$$

The structure arrow $l \circ T\eta^S_A \circ T(l \circ \eta^T_{SA})$ is equal to l because the following diagram

$$
\begin{tikzcd}
TSA \arrow[rr, "T\eta^T_{SA}"] \arrow[d, "id"'] \arrow[dr, phantom, "(1)"] & & T^2SA \arrow[rr, "Tl"] \arrow[dl, "id"'] \arrow[d, "T^2\eta^S_{SA}"] \arrow[dr, "T\eta^S_{TSA}"] & \arrow[d, phantom, "(2)"] & TA \arrow[dd, "T\eta^S_A"] \\
TSA \arrow[d, "l"'] & T^2SA \arrow[l, "\mu^T_{SA}"] \arrow[d, phantom, "(3)"] & T^2S^2A \arrow[l, "T^2\mu^S_A"] & TSTSA \arrow[l, "T\delta_{SA}"] \arrow[dr, "TSl"] & \\
A & & & & TSA \arrow[llll, "l"]
\end{tikzcd}
$$

commutes by the left unit law for the monad T (1), the naturality of η^S (2), the right unit law for the monad S (the triangle on the left), the unit law for η^S of the distributive law δ, and one of the axioms (3.14) for the TS-algebra l (3).

Again, it is easy to see that the arrow part of $K \circ K^{-1}$ is an identity. This concludes the proof of the proposition. □

From what we have seen so far, we can state the following theorem:

Theorem 3.35. *There is a canonical isomorphism between TS-Alg and* $\Xi(\delta)$*-Alg.*

Proof. Follows immediately from Proposition 3.32 and Lemma 3.34. □

We summarise the above discussion on the composite TS in the following theorem:

Theorem 3.36. *Given a distributive law $\delta : ST \to TS$ of a monad (S, μ^S, η^S) over a monad (T, μ^T, η^T) on a ordinary category $\mathcal{C}$,*

1. *the composite functor TS acquires the structure for a monad on $\mathcal{C}$, with multiplication given by*

$$TSTS \xrightarrow{T\delta S} TTSS \xrightarrow{\mu^T \mu^S} TS$$

2. *TS-Alg is canonically isomorphic to $\Xi(\delta)$-Alg*

3. *the object $TS1$ has both canonical S-algebra and T-algebra structures on it.*

As we will see in the later chapters, the results in this section all extend without fuss to the pseudo setting.

Chapter 4

Kleisli Category and Distributive Laws

Just as in the previous chapter, we consider here the situation where there exist both a monad (T, μ^T, η^T) and an endofunctor H, which we later upgrade to a monad, on a category $\mathbb{C}$. In Chapter 3 we fixed a monad and investigated distributive laws of the monad over endofunctors (or monads), and how they are related to liftings of the endofunctors (or the monads) to the category of algebras. In this chapter, we again fix the monad T but consider distributive laws over it and how they relate to the Kleisli category $Kl(T)$ of T. It is shown that there exists an isomorphism between them just as in the case of distributive laws and liftings.

This chapter mirrors the structure of Chapter 3: the main difference is that, in this chapter, instead of distributive laws of a monad S over endofunctors, we study those of endofunctors over a monad T; instead of the category of S-algebras (Eilenberg-Moore category), we consider the Kleisli category $Kl(T)$ of T; apart from these the discussion takes the same path.

We start from the definition of a distributive law of an endofunctor H over a monad T and the notion of a natural transformation distributing over T. We show that they form the category Dist_T of distributive laws over T (Section 4.1).

In Section 4.2 we study the extensions of endofunctors to the Kleisli category $Kl(T)$ of T. After recalling the definition of the Kleisli category $Kl(T)$ of a monad T, we study some of its basic properties, in particular the adjunction between $\mathbb{C}$ and $Kl(T)$. The counit of this adjunction plays an important role in the rest of the chapter. Then we give the definitions of an *extension* of an endofunctor H to $Kl(T)$ and of the notion of a natural transformation extending to $Kl(T)$. Again, they form the category $\mathsf{Ext}_{Kl(T)}$ of extensions of endofunctors to $Kl(T)$.

In the following three sections we prove the categories Dist_T and $\mathsf{Ext}_{Kl(T)}$ are isomorphic: in Section 4.3 we define a functor Υ from $\mathsf{Ext}_{Kl(T)}$ to Dist_T and in Section 4.4 a functor Γ from Dist_T to $\mathsf{Ext}_{Kl(T)}$. Then in Section 4.5 we prove that they are mutually inverse and define an isomorphism of categories between Dist_T and $\mathsf{Ext}_{Kl(T)}$.

In Section 4.6, as preparation for the monad case, we prove some properties regarding the square of an endofunctor. This section corresponds to Section 3.6 in the discussion of distributive laws and liftings.

Finally, we consider the case of distributive laws of monads over T in Section 4.7 and 4.8. After defining extensions of monads to $Kl(T)$ in Section 4.7, we prove that Υ and Γ induce functors between $\mathsf{Dist}_\mathsf{T}^{\mathsf{monads}}$ and $\mathsf{Ext}_{Kl(T)}^{\mathsf{monads}}$ determined by monads and define an isomorphism of categories in Section 4.8.

4.1 Distributivity over a monad *T*

In this section we define the notion of a distributive law of an endofunctor H over a monad T. We have, in the previous chapter, seen the definition of a distributive law of a monad S over an endofunctor, and the following definition is a natural dual.

Definition 4.1. Given an endofunctor H and a monad (T,μ,η) on a category $\mathbb{C}$, a distributive law of H over T is a natural transformation $\delta : HT \to TH$ such that the following diagrams commute:

$$\begin{tikzcd}
HT^2 \arrow[r, "\delta T"] \arrow[d, "H\mu^T"'] & HTH \arrow[r, "T\delta"] & T^2H \arrow[d, "\mu^T H"] \\
HT \arrow[rr, "\delta"'] & & TH
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[d, "H\eta^T"'] \arrow[dr, "\eta^T H"] & \\
HT \arrow[r, "\delta"'] & TH
\end{tikzcd}
\tag{4.1}$$

Similarly to the case of distributive laws of a monad over endofunctors, in the presence of two distributive laws $\delta^H : HT \to TH$ and $\delta^K : KT \to TK$, given a natural transformation $\alpha : H \to K$, we can consider the situation where α distributes over T with respect to these distributive laws.

Definition 4.2. Let $\delta^H : HT \to TH$ and $\delta^K : KT \to TK$ be distributive laws of H and K over T, respectively. Then a natural transformation $\alpha : H \to K$ *distributes over* T *with respect to* δ^H *and* δ^K if the following diagram

$$\begin{tikzcd} HT \arrow[r, "\delta^H"] \arrow[d, "\alpha T"'] & TH \arrow[d, "T\alpha"] \\ KT \arrow[r, "\delta^K"'] & TK \end{tikzcd} \tag{4.2}$$

commutes.

Proposition 4.3. *Given a monad (T,μ,η) on a category $\mathbb{C}$, the following data form a category we denote by* Dist_T*: objects of* Dist_T *are pairs $(H,\delta : HT \to TH)$ of an endofunctor H on $\mathbb{C}$ and its distributive law over T, and an arrow from (H,δ^H) and (K,δ^K) is given by a natural transformation $\alpha : H \to K$ that distributes over T with respect to δ^H and δ^K. The composition of arrows is given by composition of natural transformations.*

Notation 4.4. Just as before, we often omit the first component in the objects whenever it does not cause confusion and just write δ^H instead of $(H,\delta : HT \to TH)$.

4.2 Extension of *S* to the Kleisli Category

In this section, we turn to the Kleisli category $Kl(T)$ of the monad T. After recalling the definition of Kleisli category, we have a close look at the well-known adjunction (J,G,η,ε) between $\mathbb{C}$ and $Kl(T)$. We examine the naturality of the counit ε of this adjunction as it plays an important role in the following sections. We also state two equations on the adjoint arrows, derived from the adjunction.

4.2.1 Some properties of Kleisli categories

Definition 4.5. Let (T,μ,η) be a monad on a category $\mathbb{C}$. The *Kleisli category $Kl(T)$* of T is a category that has, for each object A in $\mathbb{C}$, an object A_T, and, for each arrow

f of type $A \to TB$ in $\mathbb{C}$, an arrow h of type $A_T \to B_T$ in $Kl(T)$. We use the notation $h = f^\flat$ and $f = h^\sharp$ to express this relation between arrows. The composition in $Kl(T)$ is defined in terms of that in $\mathbb{C}$, as follows: let $h : A_T \to B_T$ and $k : B_T \to C_T$ be arrows in $Kl(T)$. Then $k \circ h$ is defined as

$$k \circ h = (\mu_{C_T} \circ Tk^\sharp \circ h^\sharp)^\flat$$

yielding an arrow of type $A_T \to C_T$ in $Kl(T)$. It is straightforward to see that this construction is associative. The identity is given as $id_{A_T} = \eta_A^\flat : A_T \to A_T$, which is the left and right unit for the composition due to the unit law of the monad T.

As is well-known, there is a canonical adjunction between $\mathbb{C}$ and $Kl(T)$, for which the mapping between adjoint arrows is in fact given by $-^\flat$ and $-^\sharp$.

Proposition 4.6 (Another monad induced adjunction (Kleisli)). *Given a monad (T, μ, η) on $\mathbb{C}$, there is an adjunction*

$$(J, G, \eta, \varepsilon) : \mathbb{C} \underset{G}{\overset{J}{\rightleftarrows}} Kl(T)$$

where J and G are functors such that $JA = A_T$ and $GB_T = TB$ on objects A in $\mathbb{C}$ and B_T in $Kl(T)$, and for arrows $f : A \to B$ in $\mathbb{C}$ and $h : A_T \to B_T$ in $Kl(T)$,

$$\begin{aligned} Jf &= (\eta_B \circ f)^\flat : A_T \longrightarrow B_T \\ Gh &= \mu_B \circ Th^\sharp : TA \xrightarrow{Th^\sharp} T^2B \xrightarrow{\mu_B} TB. \end{aligned}$$

Note that $GJ = T$ holds not only on objects but also on arrows. The unit $\eta : Id_{\mathbb{C}} \to GJ$ of this adjunction is provided by the unit η of the monad T, and the counit $\varepsilon : JG \to Id_{Kl(T)}$ is given by

$$\varepsilon_{A_T} = id_{TA}^\flat : (TA)_T \to A_T.$$

Now we verify that the unit ε is indeed a natural transformation. For an arrow $h : A_T \to B_T$ in $Kl(T)$, the naturality of ε is given by the commutativity of the following square in $Kl(T)$

$$\begin{tikzcd}
(TA)_T \arrow[r, "\varepsilon_{A_T} = id_{TA}^\flat"] \arrow[d, "JGh"'] & A_T \arrow[d, "h"] \\
(TB)_T \arrow[r, "\varepsilon_{B_T} = id_{TB}^\flat"'] & B_T
\end{tikzcd}$$

where the composite arrows are calculated as

$$\begin{aligned} h \circ \varepsilon_{A_T} &= (\mu_B \circ Th^\sharp \circ id_{TA})^\flat \\ &= (\mu_B \circ Tid_{TB} \circ Th^\sharp)^\flat \\ &= id_{TB}^\flat \circ (Th^\sharp)^\flat. \end{aligned}$$

Since $JGh = (\eta_{TB} \circ \mu_B \circ Th^\sharp)^\flat = (\mu_{TB} \circ T\eta_{TB} \circ Th^\sharp)^\flat = \eta_{TB}^\flat \circ (Th^\sharp)^\flat = (Th^\sharp)^\flat$ holds, this demonstrates the naturality of ε.

Then there exist bijections

$$\mathbb{C}(A, GB_T) = \mathbb{C}(A, TB) \underset{\sharp}{\overset{\flat}{\rightleftarrows}} Kl(T)(A_T, B_T) = Kl(T)(JA, B_T)$$

$$\frac{A \xrightarrow{f} TB}{A_T \xrightarrow[f^\flat]{} B_T} \qquad \frac{A \xrightarrow{h^\sharp} TB}{A_T \xrightarrow[h]{} B_T}$$

which are natural in A and B_T. The naturality of these bijections amount to the following equations: for arrows $f : A \to TB$ and $g : B \to TB'$ in $Kl(T)$ and $m : A' \to A$ in $\mathbb{C}$,

$$g^\flat \circ f^\flat = (Gg^\flat \circ f)^\flat \tag{4.3a}$$

$$(f \circ m)^\flat = f^\flat \circ Jm, \tag{4.3b}$$

and two more equations involving only $\sharp$, which are inverses of these. Note the first equation (4.3a) gives the definition of composition of arrows in $Kl(T)$.

4.2.2 Extension to *Kl(T)*

Here we consider the situation where there exist both a monad (T, μ^T, η^T) and an endofunctor H, which we later upgrade to a monad, on a category $\mathbb{C}$. We define the notion of an *extension* of H to $Kl(T)$, which is an endofunctor on $Kl(T)$ that commutes with the left adjoint J and H. We then state a useful lemma which gives the values of an extension at adjoint arrows. After considering the extensions of natural transformations and some of their properties, we give the definition of extensions of monads on $\mathbb{C}$ to *Kl(T)*, which is the second of the two main ingredients of the later discussion in this section.

Definition 4.7. Given a monad (T,μ^T,η^T) and an endofunctor H on a category $\mathbb{C}$, an *extension* of H to $Kl(T)$ is an endofunctor $\widetilde{H}$ on $Kl(T)$, for which $\widetilde{H}J = JH$ holds.

The condition requires the functor $\widetilde{H}$ to satisfy $\widetilde{H}A_T = (HA)_T$ on an object A_T, and for an arrow $m : A \to B$ in $\mathbb{C}$,

$$\widetilde{H}(\eta^T_B \circ m)^\flat = (\eta^T_{HB} \circ Hm)^\flat. \tag{4.4}$$

Since $\widetilde{H}$ is a functor, we have equalities $\widetilde{H}(k \circ h) = \widetilde{H}k \circ \widetilde{H}h$ and $\widetilde{H}id_A = id_{\widetilde{H}A_T}$, where $h : A_T \to B_T$ and $k : B_T \to C_T$ are arrows in $\mathbb{C}$. The first of the two equations amounts to the equality

$$(\mu^T_{HC} \circ T(\widetilde{H}k)^\sharp \circ (\widetilde{H}h)^\sharp)^\flat = \widetilde{H}(\mu^T_C \circ Tk^\sharp \circ h^\sharp)^\flat. \tag{4.5}$$

And, since the identity arrow $id_{A_T} : A_T \to A_T$ in $Kl(T)$ is $\eta^{T\flat}_A$, the second equation gives the equality

$$\widetilde{H}\eta^{T\flat}_A = \eta^{T\ \flat}_{HA}. \tag{4.6}$$

Moreover, we have the following lemma:

Lemma 4.8. *For any arrow $h : A_T \to B_T$ in $Kl(T)$*

$$\widetilde{H}h = ((\widetilde{H}\varepsilon_{B_T})^\sharp \circ Hh^\sharp)^\flat \tag{4.7}$$

holds.

Proof. Recall $\varepsilon_{B_T} = id^\flat_{TB}$. Then, the right hand side calculates:

$$\begin{aligned}
((\widetilde{H}id^\flat_{TB})^\sharp \circ Hh^\sharp)^\flat &= \widetilde{H}id^\flat_{TB} \circ JHh^\sharp && \text{by (4.3b)}\\
&= \widetilde{H}id^\flat_{TB} \circ \widetilde{H}Jh^\sharp\\
&= \widetilde{H}(\mu^T_B \circ Tid_{TB} \circ \eta^T_B \circ h^\sharp)^\flat\\
&= \widetilde{H}h
\end{aligned}$$

□

The notion of extension to *Kl(T)* naturally extends to natural transformations.

Definition 4.9. Given extensions $\widetilde{H}$, $\widetilde{K}$ of endofunctors H and K, respectively, and a natural transformation $\alpha : H \to K$ on $\mathbb{C}$, we say "α *extends to* $Kl(T)$ *from* $\widetilde{H}$ *to* $\widetilde{K}$" if, for each A in $\mathbb{C}$, $J\alpha_A$ is the component at A_T of a natural transformation in $Kl(T)$, or equivalently, there exists a natural transformation $\widetilde{\alpha} : \widetilde{H} \to \widetilde{K}$, such that $\widetilde{\alpha}J = J\alpha$ holds.

Then, from the naturality condition α for the components of the counit ε of the adjunction, we have the following lemma to characterise this notion:

Lemma 4.10. *Given extensions $\widetilde{H}$, $\widetilde{K}$ of endofunctors H and K, a natural transformation $\alpha : H \to K$ extends to $Kl(T)$ from $\widetilde{H}$ to $\widetilde{K}$ if and only if for each A the following holds:*

$$(\widetilde{K}\varepsilon_{A_T})^\sharp \circ \alpha_{TA} = GJ\alpha_A \circ (\widetilde{H}\varepsilon_{A_T})^\sharp. \tag{4.8}$$

Proof. If there exists a natural transformation $\widetilde{\alpha} : \widetilde{H} \to \widetilde{K}$ such that $\widetilde{\alpha}J = J\alpha$, then from its naturality it follows that for each component ε_{A_T} of ε, the equality

$$\widetilde{K}\varepsilon_{A_T} \circ \widetilde{\alpha}_{(TA)_T} = \widetilde{\alpha}_{A_T} \circ \widetilde{H}\varepsilon_{A_T}$$

should hold. The adjoint of the left hand side is an arrow $(\widetilde{K}\varepsilon_{A_T})^\sharp \circ \alpha_{TA}$ in $\mathbb{C}$, which equals the adjoint of the right hand side $GJ\alpha_A \circ (\widetilde{H}\varepsilon_{A_T})^\sharp$, proving that the equality (4.8) holds.

For the opposite direction, assume the equality (4.8) holds for α. Then, for any arrow $h : A_T \to B_T$ in $\mathbb{C}$, one can calculate

$$\begin{aligned}
\widetilde{K}h \circ J\alpha_A &= ((\widetilde{K}\varepsilon_{B_T})^\sharp \circ Kh^\sharp)^\flat \circ J\alpha_A \\
&= ((\widetilde{K}\varepsilon_{B_T})^\sharp \circ Kh^\sharp \circ \alpha_A)^\flat \\
&= ((\widetilde{K}\varepsilon_{B_T})^\sharp \circ \alpha_{TB} \circ Hh^\sharp)^\flat \\
&= (GJ\alpha_B \circ (\widetilde{H}\varepsilon_{B_T})^\sharp \circ Hh^\sharp)^\flat \\
&= J\alpha_B \circ ((\widetilde{H}\varepsilon_{B_T})^\sharp \circ Hh^\sharp)^\flat \\
&= J\alpha_B \circ \widetilde{H}h,
\end{aligned}$$

proving the commutativity of the naturality square

$$\begin{tikzcd}
(HA)_T \arrow[r, "J\alpha_A"] \arrow[d, "\widetilde{H}h"'] & (KA)_T \arrow[d, "\widetilde{K}h"] \\
(HB)_T \arrow[r, "J\alpha_B"'] & (KB)_T .
\end{tikzcd}$$

□

Extensions of endofunctors on and natural transformations between them form a category.

Proposition 4.11. *Let (T,μ,η) be a monad on a category $\mathbb{C}$. The following data yields a category we denote by $\mathsf{Ext}_{Kl(T)}$: objects $(H,\widetilde{H})$ are pairs of an endofunctor H on $\mathbb{C}$ and its extension $\widetilde{H}$ to $Kl(T)$, and an arrow from $(H,\widetilde{H})$ and $(K,\widetilde{K})$ is given by a natural transformation $\alpha : H \to K$ that extends to $Kl(T)$ from $\widetilde{H}$ to $\widetilde{K}$. The composition of arrows is given by that of natural transformations.*

4.3 From extensions to distributive laws

Now we can prove the following proposition.

Proposition 4.12. *Given a monad (T,μ^T,η^T) and an endofunctor H on $\mathbb{C}$, an extension $\widetilde{H}$ of H to $Kl(T)$ gives rise to a distributive law from HT to TH in $\mathbb{C}$.*

Proof. We define a mapping Υ^H from extensions of H to $Kl(T)$ to distributive laws of H over T. The value of Υ at $\widetilde{H}$ is a natural transformation $\Upsilon(\widetilde{H}) : HT \to TH$, whose component $\Upsilon(\widetilde{H})_A$ of $\Upsilon(\widetilde{H})$ at A is defined to be the arrow $(\widetilde{H}\varepsilon_{A_T})^\sharp$ in $\mathbb{C}$. We need to prove that this is a distributive law. First, we need to see that it is a natural transformation. The naturality square for an arrow $m : A \to B$ in $\mathbb{C}$ is

$$\begin{tikzcd}
HTA \arrow[r, "(\widetilde{H}\varepsilon_{A_T})^\sharp"] \arrow[d, "HTm"'] & THA \arrow[d, "THm"] \\
HTB \arrow[r, "(\widetilde{H}\varepsilon_{B_T})^\sharp"'] & THB
\end{tikzcd}$$

for which a proof of commutativity is given as follows: the composite of the top and the right sides are

$$\begin{aligned} THm \circ (\widetilde{H}\varepsilon_{A_T})^\sharp &= GJHm \circ (\widetilde{H}\varepsilon_{A_T})^\sharp \\ &= G\widetilde{H}Jm \circ (\widetilde{H}\varepsilon_{A_T})^\sharp \\ &= (\widetilde{H}Jm \circ \widetilde{H}\varepsilon_{A_T})^\sharp \\ &= (\widetilde{H}\varepsilon_{B_T} \circ \widetilde{H}JGJm)^\sharp \\ &= (\widetilde{H}\varepsilon_{B_T} \circ JHGJm)^\sharp \\ &= (\widetilde{H}\varepsilon_{B_T})^\sharp \circ HTm, \end{aligned}$$

proving the commutativity of the square above.

Next, to see that our choice of distributive law satisfies the associative and the unit laws, we examine the two axioms. For the associative law of $\Upsilon(\widetilde{H})$ and μ^T,

$$\begin{tikzcd} HT^2A \arrow[r, "(\widetilde{H}\varepsilon_{(TA)_T})^\sharp"] \arrow[d, "H\mu^T_A"'] & THTA \arrow[r, "T(\widetilde{H}\varepsilon_{A_T})^\sharp"] & T^2HA \arrow[d, "\mu^T_{HA}"] \\ HTA \arrow[rr, "(\widetilde{H}\varepsilon_{A_T})^\sharp"'] & & THA \end{tikzcd}$$

This commutes simply by the definition of composition in $Kl(T)$. The top and the right sides compose as

$$\begin{aligned} \mu^T_{HA} \circ T(\widetilde{H}\varepsilon_{A_T})^\sharp \circ (\widetilde{H}\varepsilon_{(TA)_T})^\sharp &= (\widetilde{H}(\mu^T_A \circ Tid_{TA} \circ id_{T^2A})^\flat)^\sharp \\ &= (\widetilde{H}(\mu^T_A)^\flat)^\sharp \end{aligned}$$

which is equal to $(\widetilde{H}\varepsilon_{A_T})^\sharp \circ H\mu^T_A$ from the Lemma 4.8, proving the commutativity of the diagram. Finally, the unit law of $\Upsilon(\widetilde{H})$ and η^T is

$$\begin{tikzcd} HA \arrow[d, "H\eta^T_A"'] \arrow[dr, "\eta^T_{HA}"] & \\ HTA \arrow[r, "(\widetilde{H}id^\flat_{TA})^\sharp"'] & THA \end{tikzcd}$$

whose commutativity follows immediately from Lemma 4.8 and the equation (4.6). This concludes the proof of Proposition 4.12. ☐

Corollary 4.13. *The mapping Υ in Proposition 4.12 defines a functor Υ from $\mathsf{Ext}_{Kl(T)}$ to Dist_T.*

Proof. Define $\Upsilon(H,\widetilde{H}) = \Upsilon(\widetilde{H})$ and $\Upsilon(\alpha) = \alpha : \Upsilon(\widetilde{H}) \to \Upsilon(\widetilde{K})$ for any $\alpha : \widetilde{H} \to \widetilde{K}$. The definition of the arrow part is justified by Lemma 4.10. The condition 4.8 makes the diagram 3.1 hold for $\alpha : \Upsilon(\widetilde{H}) \to \Upsilon(\widetilde{K})$. □

4.4 From distributive laws to extensions

Proposition 4.14. *A distributive law $\delta : HT \to TH$ of an endofunctor H over a monad (T,μ,η) gives rise to an extension of H to $Kl(T)$.*

Proof. We define a mapping Γ^H from distributive laws of H over T to extensions of H to $Kl(T)$. Define $\Gamma^H(\delta)$ to be an endofunctor on $Kl(T)$ such that the values at an object A_T and an arrow $h : A_T \to B_T$ of $Kl(T)$ are given as $\Gamma^H(\delta)A_T = (HA)_T$ and $\Gamma^H(\delta)h = (\delta_B \circ Hh^\sharp)^\flat$. We verify the functoriality of $\Gamma^H(\delta)$: both conditions on the composition and the identity hold because of the two axioms of δ. For the composition, given a pair of arrows $h : A_T \to B_T$ and $k : B_T \to C_T$ in $Kl(T)$, we have

$$\begin{aligned}
\Gamma^H(\delta)k \circ \Gamma^H(\delta)h &= (\delta_C \circ Hk^\sharp)^\flat \circ (\delta_B \circ Hh^\sharp)^\flat \\
&= (G(\delta_C \circ Hk^\sharp)^\flat \circ \delta_B \circ Hh^\sharp)^\flat \\
&= (\mu_{HC} \circ T\delta_C \circ THk^\sharp \circ \delta_B \circ Hh^\sharp)^\flat \\
&= (\mu_{HC} \circ T\delta_C \circ \delta_{TC} \circ HTk^\sharp \circ Hh^\sharp)^\flat \\
&= (\delta_C \circ H\mu_C \circ HTk^\sharp \circ Hh^\sharp)^\flat \\
&= (\delta_C \circ H(k \circ h)^\sharp)^\flat = \Gamma^H(\delta)(k \circ h)
\end{aligned}$$

and for the identity,

$$\Gamma^H(\delta)id_{A_T} = (\delta_A \circ H\eta_A)^\flat = \eta^\flat_{HA}.$$

Finally we prove that $\Gamma^H(\delta)$ is an extension of H to $Kl(T)$. It obviously satisfies the condition on objects. For arrows, given any arrow $m : A \to B$ in $\mathbb{C}$, we have

$$\Gamma^H(\delta)Jm = \Gamma^H(\delta)(\eta_B \circ m)^\flat = (\delta_B \circ H\eta_B \circ Hm)^\flat = (\eta_{HB} \circ Hm)^\flat$$

as desired. This completes the proof of Proposition 4.14. □

Corollary 4.15. *The mapping* Γ *in Proposition 4.14 defines a functor* Γ *from* Dist_T *to* $\mathsf{Ext}_{Kl(T)}$.

Proof. Define $\Gamma(H, \delta^H : HT \to TH) = \Gamma^H(\delta^H)$ and $\Gamma(\alpha) = \alpha : \Gamma(\delta^H) \to \Gamma(\delta^K)$ for any $\alpha : \delta^H \to \delta^K$. Justification for the definition of the arrow part is similar to that in the definition of Υ (Corollary 4.13, by Lemma 4.10.) $\square$

4.5 Isomorphism between Dist_T and $\mathsf{Ext}_{Kl(T)}$

Theorem 4.16. *The functors* Υ *and* Γ *are mutually inverse and, indeed, define an isomorphism of the categories* $\mathsf{Ext}_{Kl(T)}$ *and* Dist_T.

Proof. The constructions shown in the previous two sections are mutually inverse. Given an extension $\widetilde{H}$, the induced distributive law $\Upsilon(\widetilde{H})$ defines an extension $\Gamma(\Upsilon(\widetilde{H}))$ that sends $h : A_T \to B_T$ to

$$(\Upsilon(\widetilde{H})_B \circ Hh^\sharp)^\flat = ((\widetilde{H}\varepsilon_{B_T})^\sharp \circ Hh^\sharp)^\flat = \widetilde{H}h,$$

which is the same arrow as the one $\widetilde{H}$ sends h to, meaning they define the same extension. Meanwhile, given a distributive law δ, the induced extension $\Gamma(\delta)$ defines a distributive law $\Upsilon(\Gamma(\delta))$, whose component at A is given by

$$(\Gamma(\delta)\varepsilon_{A_T})^\sharp = \delta_A \circ H\varepsilon^\sharp_{A_T} = \delta_A \circ Hid_{TA} = \delta_A$$

proving that it defines the original distributive law δ. This completes the proof of the theorem. $\square$

4.6 Extension of H^2

In this section, we consider distributive laws and extensions of the endofunctor H^2, along the lines of Section 3.6. The discussion in this section can be formulated in terms of strict monoidal structures on $\mathsf{Ext}_{Kl(T)}$ and Dist_T, and it extends to the pseudo-case in a similar manner as in Chapter 6 for the liftings.

Given an extension $\widetilde{H}$ of an endofunctor H to $Kl(T)$, it is immediate that the composite $\widetilde{H}^2$ is an extension of H^2. On the other hand, given a distributive law

$\delta : HT \to TH$ of H over T, we have the following lemma to construct a distributive law of H^2:

Lemma 4.17. *Given a distributive law $\delta : HT \to TH$ of an endofunctor H and a monad T on a category $\mathbb{C}$, the natural transformation*

$$HHT \xrightarrow{H\delta} HTH \xrightarrow{\delta H} THH$$

is a distributive law of H^2 over T.

Proof. By a similar discussion as Lemma 3.20. □

The following proposition relates the functors Υ and Γ to the above discussion.

Proposition 4.18. *Given an endofunctor H on $\mathbb{C}$ and an extension $\widetilde{H}$ of H to $Kl(T)$ and a distributive law δ of H over T,*

$$\widetilde{H}^2 = \Gamma(\Upsilon(\widetilde{H})H \circ H\Upsilon(\widetilde{H})) \tag{4.9}$$

$$\delta H \circ H\delta = \Upsilon(\Gamma(\delta)^2) \tag{4.10}$$

hold.

Proof. Immediate from the definitions of Υ and Γ. □

Similarly to the case for liftings, the isomorphisms Υ and Γ are both strict monoidal functors.

4.7 Categories $\mathsf{Dist}_{\mathsf{T}}^{\mathsf{monads}}$ and $\mathsf{Ext}_{Kl(T)}^{\mathsf{monads}}$

In the rest of the chapter we consider extensions of monads to $Kl(T)$ and study their relationship to distributive laws of monads over T. In order to do so, we first define the variants of $\mathsf{Dist}_{\mathsf{T}}$ and $\mathsf{Ext}_{Kl(T)}$, which we denote by $\mathsf{Dist}_{\mathsf{T}}^{\mathsf{monads}}$ and $\mathsf{Ext}_{Kl(T)}^{\mathsf{monads}}$.

The category $\mathsf{Dist}_{\mathsf{T}}^{\mathsf{monads}}$ is very similar to $\mathsf{Dist}_{\mathsf{monads}}^{\mathsf{S}}$ defined in Section 3.7.

Proposition 4.19. *The following data form a category we denote by $\mathsf{Dist}_{\mathsf{T}}^{\mathsf{monads}}$: objects are pairs $((S,\mu^S,\eta^S),\delta : ST \to TS)$ of a monad (S,μ^S,η^S) and a distributive law δ of S over T, and an arrow from $((S,\mu^S,\eta^S),\delta^S : ST \to TS)$ to $((S',\mu^{S'},\eta^{S'}),\delta^{S'} : S'T \to TS')$ is given by a monad morphism $\alpha : (S,\mu^S,\eta^S) \to (S',\mu^{S'},\eta^{S'})$ that distributes over T with respect to δ^S and $\delta^{S'}$.*

We now give the definition of an extension of a monad S:

Definition 4.20. Given monads (S,μ^S,η^S) and (T,μ^T,η^T) on a category $\mathbb{C}$, an *extension* of S to $Kl(T)$ is a monad $(\widetilde{S},\mu^{\widetilde{S}},\eta^{\widetilde{S}})$, for which $\widetilde{S}J = JS$ holds and μ^S and η^S extend to $Kl(T)$, meaning $\mu^{\widetilde{S}}J = J\mu^S$ and $\eta^{\widetilde{S}}J = J\eta^S$ hold.

The first condition means, just as the case of an endofunctor H, $\widetilde{S}$ satisfies $\widetilde{S}A_T = (SA)_T$ on an object A_T, and $\widetilde{S}(\eta^T_B \circ m)^\flat = (\eta^T_{SB} \circ Sm)^\flat$ holds for an arrow $m : A \to B$ in $\mathbb{C}$.

And, from Lemma 4.10, the naturality of $\mu^{\widetilde{S}}$ and $\eta^{\widetilde{S}}$ implies that $\widetilde{S}$ is an endofunctor that makes the following equations hold:

$$(\widetilde{S}\varepsilon_{A_T})^\sharp \circ \mu^S_{TA} = GJ\mu^S_A \circ (\widetilde{S}^2\varepsilon_{A_T})^\sharp \tag{4.11}$$

$$(\widetilde{S}\varepsilon_{A_T})^\sharp \circ \eta^S_{TA} = GJ\eta^S_A \tag{4.12}$$

Just as in the case of liftings of monad morphisms, we have the following lemma:

Lemma 4.21. *Given a monad morphism $\alpha : (S,\mu^S,\eta^S) \to (S',\mu^{S'},\eta^{S'})$, if it extends to $Kl(T)$, then its extension $\widetilde{\alpha}$ is a monad morphism from $(\widetilde{S},\mu^{\widetilde{S}},\eta^{\widetilde{S}})$ to $(\widetilde{S}',\mu^{\widetilde{S}'},\eta^{\widetilde{S}'})$.*

Hence we can construct a category as follows:

Proposition 4.22. *The following data form a category we denote by $\mathsf{Ext}^{monads}_{Kl(T)}$: objects are pairs $((S,\mu^S,\eta^S),\widetilde{S})$ of a monad (S,μ^S,η^S) and its extension $\widetilde{S}$ as a monad to $Kl(T)$, and an arrow from $((S,\mu^S,\eta^S),\widetilde{S})$ to $((S',\mu^{S'},\eta^{S'}),\widetilde{S}')$ is given by a monad morphism $\alpha : (S,\mu^S,\eta^S) \to (S',\mu^{S'},\eta^{S'})$ that extends to $Kl(T)$ from $\widetilde{S}$ to $\widetilde{S}'$.*

4.8 Restricting isomorphisms

In the following, we prove that the isomorphisms Υ and Γ induce functors between these categories and define an isomorphism between them.

Proposition 4.23. *The functor Υ in Proposition 4.12 induces a functor from $\mathsf{Ext}^{monads}_{Kl(T)}$ to Dist^{monads}_T.*

Proof. Define $\Upsilon((S,\mu^S,\eta^S),\widetilde{S}) = ((S,\mu^S,\eta^S),\Upsilon(\widetilde{S}))$ and $\Upsilon(\alpha) = \alpha$. We show that the distributive law $\Upsilon(\widetilde{S})$ is that of S as a monad over T, by examining the two additional axioms for μ^S and η^S. For the multiplication law of δ and μ^S, recall that μ^S extends to $\mu^{\widetilde{S}}$, therefore the equation (4.8) holds for μ^S, which is

$$(\widetilde{S}\varepsilon_{A_T})^\sharp \circ \mu^S_{TA} = T\mu^S_A \circ (\widetilde{S}^2\varepsilon_{A_T})^\sharp.$$

On the other hand, from Lemma 4.8 we have the equality

$$(\widetilde{S}^2\varepsilon_{A_T})^\sharp = (\widetilde{S}\varepsilon_{(SA)_T})^\sharp \circ S(\widetilde{S}\varepsilon_{A_T})^\sharp.$$

Putting the two equations together, we have the following diagram commutes:

$$\begin{tikzcd}
S^2TA \arrow[r, "S(\widetilde{S}\varepsilon_{A_T})^\sharp"] \arrow[d, "\mu^S_{TA}"'] & STSA \arrow[r, "(\widetilde{S}\varepsilon_{(SA)_T})^\sharp"] & TS^2A \arrow[d, "T\mu^S_A"] \\
STA \arrow[rr, "(\widetilde{S}\varepsilon_{A_T})^\sharp"'] & & TSA
\end{tikzcd}$$

For the unit law of $\delta^{\widetilde{S}}$ and η^S, a similar discussion as above for η^S holds. Since η^S extends to $Kl(T)$, the equation (4.8)

$$(\widetilde{S}\varepsilon_{A_T})^\sharp \circ \eta^S_{TA} = T\eta^S_A \circ (\varepsilon_{A_T})^\sharp$$

holds. By definition, $(\varepsilon_{A_T})^\sharp = id_{TA}$; therefore it amounts to the commutativity of

$$\begin{tikzcd}
TA \arrow[d, "\eta^S_{TA}"'] \arrow[dr, "T\eta^S_A"] & \\
STA \arrow[r, "(\widetilde{S}\varepsilon_{A_T})^\sharp"'] & TSA
\end{tikzcd}$$

This concludes the proof of Proposition 4.23. □

Proposition 4.24. *The functor Γ in Proposition 4.14 induces a functor from Dist^{monads}_T to $\mathsf{Ext}^{monads}_{Kl(T)}$.*

Proof. Define $\Gamma((S,\mu^S,\eta^S),\delta^S : ST \to TS) = ((S,\mu^S,\eta^S),\Gamma(\delta^S))$ We show that the extension $\Gamma(\delta^S)$ of S to $Kl(T)$ is a monad, i.e., μ^S and η^S also extend to $Kl(T)$. Using

Lemma 4.10, we only need to prove the following equalities: for μ^S,

$$\begin{aligned}(\Gamma(\delta^S)\varepsilon_{A_T})^\sharp \circ \mu^S_{TA} &= \delta_A \circ Sid_{TA} \circ \mu^S_{TA} \\ &= \delta_A \circ \mu^S_{TA} \\ &= T\mu^S_A \circ \delta_{SA} \circ S\delta_A \circ Sid_{TA} \\ &= T\mu^S_A \circ ((\Gamma(\delta^S))^2 \varepsilon_{A_T})^\sharp,\end{aligned}$$

and for η^S,

$$\begin{aligned}(\Gamma(\delta^S)\varepsilon_{A_T})^\sharp \circ \eta^S_{TA} &= \delta_A \circ \eta^S_{TA} \\ &= T\eta^S_A,\end{aligned}$$

This defines a monad $(\Gamma(\delta^S), \mu^{\Gamma(\delta^S)}, \eta^{\Gamma(\delta^S)})$. This completes the proof of Proposition 4.24. $\square$

From the above two proposition and Theorem 4.16, we have the following:

Corollary 4.25. *The categories* $\mathsf{Ext}^{monads}_{Kl(T)}$ *and* Dist^{monads}_T *are isomorphic.*

4.9 Discussion

We state the following theorem as a summary of Chapters 3 and 4 for distributive laws of a monad over a monad:

Theorem 4.26. *Let* (S, μ^S, η^S) *and* (T, μ^T, η^T) *be monads on a category* $\mathbb{C}$. *Then the following are equivalent:*

1. *a distributive law* $\delta : ST \to TS$ *of* S *over* T.

2. *a lifting of* T *to* S*-Alg*

3. *an extension of* S *to* $Kl(T)$

Chapter 5

Pseudo-Distributive Laws I

In Chapter 3 we proved that, given a monad S on a category $\mathcal{C}$, the category of distributive laws of a monad S over endofunctors is isomorphic to that of liftings of endofunctors to *S-Alg*, the category of S-algebras, and that the discussion extends naturally to the case where the endofunctors have the structure of monads. In this chapter, we further extend our discussion to the case of pseudo-distributive laws on a 2-category. We prove that, given a pseudo-monad S on a 2-category $\mathbb{C}$, the 2-category of pseudo-distributive laws of S over pseudo-endofunctors is equivalent (in the category-theoretic sense) to that of liftings of pseudo-endofunctors to the 2-category *Ps-S-Alg* of pseudo-S-algebras.

The reason we need to study pseudo-distributive laws is that in applying our theory on substitution in the later chapters, we will be dealing with composites of 2-functors, and there we need pseudo-distributive laws rather than 2-distributive laws; all the leading examples are of pseudo-distributive laws that are not strict.

In Section 5.1 we first define pseudo-distributive laws of a pseudo-monad S over pseudo-endofunctors, which we will extend to those over pseudo-monads in Chapter 7. Our discussion in principle follows the one on ordinary (non-pseudo) distributive laws, but we systematically replace the commutative diagrams with invertible 2-cells, and spell out the three coherence axioms which those 2-cells should satisfy. We also define pseudo-distributive laws over pseudo-natural transformations and consider pseudo-distributivity over modifications. In contrast to the ordinary case in Section 3.1, where the distributivity of S over a natural transformation is given as a property the natural transformation may possess, in the pseudo-case, the pseudo-distributivity of S over a pseudo-natural transformation requires additional data that is associated to the pseudo-natural transformation to make S distribute over it. On the other hand, the pseudo-distributivity of S over a modification in turn is given as a property it may satisfy. To end the section we show that the definitions in the section form a 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ of pseudo-distributive laws of S over pseudo-endofunctors.

The definition of liftings of pseudo-endofunctors to *Ps-S-Alg* is given in Section 5.2, followed by the definition of liftings of pseudo-natural transformations and modifications. Just as in the non-pseudo case (Section 3.2), a lifting of a pseudo-endofunctor H to *Ps-S-Alg* is defined to be a pseudo-endofunctor $\widehat{H}$ for which $U\widehat{H} = HU$, where U is the forgetful 2-functor from *Ps-S-Alg* to $\mathbb{C}$. Here for the pseudo-case, we have more pieces of data to define and again the commutative diagrams are replaced by invertible 2-cells. A comparison similar to that in the previous section between the pseudo and non-pseudo cases exists in the discussion of liftings. In Section 3.2, the notion of a lifting of a natural transformation is in terms of a property of the natural transformation, whereas in the pseudo-case it is an extra piece of data that is associated to the pseudo-natural transformation in a particular manner, to form a lifting. On the other hand, the lifting of a modification in turn is given as a property it may satisfy. Then we see that these form the 2-category $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ of liftings of pseudo-endofunctors to *Ps-S-Alg*.

Our goal in this chapter is to prove that there exists an equivalence of 2-categories between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$. We construct the 2-functors that provide the equivalence in two sections: in Section 5.3 we first look at the construction from $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ to $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. After proving in Section 5.3.1 a few lemmas about the properties of free pseudo-S-algebras that we need, we give the construction for each cell, that defines the 2-functor Φ from $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ to $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. Then in Section 5.4 we give the construction for the opposite direction and define the 2-functor Ψ from $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ to $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$. Finally, in Section 5.5 we prove that these 2-functors define an equivalences of 2-categories.

In the next chapter, Chapter 6, we investigate, in both the non-pseudo and pseudo cases, the square H^2 of an endofunctor H; the motivation is the same as that for Section 3.6. For the pseudo-case, we provide a more general analysis of the structure, in particular, we consider the operations constructing a lifting of H^2 from that of H and a distributive law of S over H^2 from that over H, and how those operations relate to each other. That provides the basis for the discussion in Chapter 7, where we extend the pseudo-endofunctors in the discussion of this chapter to pseudo-monads.

5.1 The 2-category Ps-Dist^S

5.1.1 Pseudo-distributive laws over pseudo-endofunctors

Recall the definition of pseudo-monads in Section 2.3. A pseudo-distributive law of a pseudo-monad $(S,\mu,\eta,\tau,\lambda,\rho)$ over a pseudo-endofunctor consists of a pseudo-natural transformation, together with two invertible modifications, one of which is the associative law involving μ and the other the unit law involving η. Both of these modifications are subject to three coherence axioms.

Definition 5.1. Given a pseudo-monad $S = (S,\mu,\eta,\tau,\lambda,\rho)$ and a pseudo-endofunctor H on a 2-category $\mathbb{C}$, a *pseudo-distributive law* $(\delta,\overline{\mu},\overline{\eta})$ of S over H consists of the following data:

- a pseudo-natural transformation

$$\delta : SH \longrightarrow HS$$

- invertible modifications $\overline{\mu}$ and $\overline{\eta}$ given as

$$\begin{tikzcd} S^2H \arrow[r, "S\delta"] \arrow[d, "\mu H"'] & SHS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow\overline{\mu}"] & HS^2 \arrow[d, "H\mu"] \\ SH \arrow[rr, "\delta"'] & & HS \end{tikzcd} \qquad \begin{tikzcd} H \arrow[d, "\eta H"'] \arrow[dr, "H\eta"] \arrow[dr, phantom, "\Uparrow\overline{\eta}"', near start] & \\ SH \arrow[r, "\delta"'] & HS \end{tikzcd} \tag{5.1}$$

In addition to the axiom for modifications, the above invertible modifications are subject to the following three coherence axioms: Axioms (H-1) and (H-2) involve the modifications λ and ρ of the pseudo-monad S. Axiom (H-3) involves the modification

τ.

(H-1)
$$
\begin{tikzcd}
 & S^2H \arrow[r, "S\delta"] \arrow[d, "\mu H"] & SHS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}"] & HS^2 \arrow[d, "H\mu"] \\
SH \arrow[ur, "S\eta H"] \arrow[r, "id_{SH}"'] \arrow[ur, phantom, "\Downarrow \lambda H"'] & SH \arrow[rr, "\delta"'] & & HS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S^2H \arrow[r, "S\delta"] & SHS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow \delta_\eta"] & HS^2 \arrow[dr, "H\mu"] \arrow[dr, phantom, "\Downarrow H\lambda"'] & \\
SH \arrow[u, "S\eta H"] \arrow[ur, "SH\eta"'] \arrow[u, phantom, "\Downarrow S\overline{\eta}"'] \arrow[rr, "\delta"'] & & HS \arrow[u, "HS\eta"'] \arrow[r, "id_{HS}"'] & HS
\end{tikzcd}
$$

(H-2)
$$
\begin{tikzcd}
 & S^2H \arrow[r, "S\delta"] \arrow[d, "\mu H"] & SHS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}"] & HS^2 \arrow[d, "H\mu"] \\
SH \arrow[ur, "\eta SH"] \arrow[r, "id_{SH}"'] \arrow[ur, phantom, "\Downarrow \rho H"'] & SH \arrow[rr, "\delta"'] & & HS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S^2H \arrow[r, "S\delta"] \arrow[d, phantom, "\Downarrow \eta_\delta"] & SHS \arrow[r, "\delta S"] & HS^2 \arrow[dr, "H\mu"] \arrow[dr, phantom, "\Downarrow H\rho"'] & \\
SH \arrow[u, "\eta SH"] \arrow[rr, "\delta"'] & & HS \arrow[ul, "\eta HS"] \arrow[u, "H\eta S"'] \arrow[u, phantom, "\Uparrow \overline{\eta} S"] \arrow[r, "id_{HS}"'] & HS
\end{tikzcd}
$$

(H-3)
$$
\begin{tikzcd}
 & S^3H \arrow[dl, "\mu SH"'] \arrow[r, "S^2\delta"] \arrow[d, "S\mu H"] & S^2HS \arrow[r, "S\delta S"] \arrow[d, phantom, "\Downarrow S\overline{\mu}"] & SHS^2 \arrow[r, "\delta S^2"] \arrow[d, "SH\mu"] & HS^3 \arrow[d, "HS\mu"] \arrow[dl, phantom, "\Downarrow \delta_\mu"] \\
S^2H \arrow[r, phantom, "\Downarrow \tau H"] \arrow[dr, "\mu H"'] & S^2H \arrow[rr, "S\delta"] \arrow[d, "\mu H"] & \arrow[d, phantom, "\Downarrow \overline{\mu}"] & SHS \arrow[r, "\delta S"] & HS^2 \arrow[d, "H\mu"] \\
 & SH \arrow[rrr, "\delta"'] & & & HS
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
S^3H \arrow[r, "S^2\delta"] \arrow[d, "\mu SH"'] \arrow[r, phantom, "\Downarrow \mu_\delta"', shift right=4] & S^2HS \arrow[r, "S\delta S"] \arrow[d, "\mu HS"] & SHS^2 \arrow[r, "\delta S^2"] \arrow[d, phantom, "\Downarrow \overline{\mu} S"] & HS^3 \arrow[d, "H\mu S"'] \arrow[dr, "HS\mu"] & \\
S^2H \arrow[r, "S\delta"] \arrow[d, "\mu H"'] & SHS \arrow[rr, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}"] & & HS^2 \arrow[r, phantom, "\Downarrow H\tau"] \arrow[d, "H\mu"'] & HS^2 \arrow[dl, "H\mu"] \\
SH \arrow[rrr, "\delta"'] & & & HS &
\end{tikzcd}
$$

5.1.2 Pseudo-distributive laws over pseudo-natural transformations

Under the existence of pseudo-distributive laws $(\delta^H, \overline{\mu}^H, \overline{\eta}^H)$ and $(\delta^K, \overline{\mu}^K, \overline{\eta}^K)$, where $\delta^H : SH \to HS$ and $\delta^K : SK \to KS$, we can consider a situation where S distributes over a pseudo-natural transformation $\alpha : H \to K$ in terms of the horizontal composition, as in

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbb{C} \arrow[r, "S"] & \mathbb{C}
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "S"] & \mathbb{C} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbb{C}.
\end{tikzcd}
$$

A transformation of this kind is realised by an invertible modification with certain properties specified in the following definition:

Definition 5.2. A *pseudo-distributive law of S over α with respect to δ^H and δ^K* is an invertible modification α^* of the following form,

$$\begin{tikzcd}
SH \arrow[r, "\delta^H"] \arrow[d, "S\alpha"'] \arrow[dr, phantom, "\Downarrow\alpha^*"] & HS \arrow[d, "\alpha S"] \\
SK \arrow[r, "\delta^K"'] & KS
\end{tikzcd}$$

satisfying, in addition to the axiom of modifications, the two axioms given below:

$$\begin{tikzcd}
S^2H \arrow[r, "S\delta^H"] \arrow[d, "\mu H"'] \arrow[dr, "S^2\alpha"] \arrow[d, phantom, shift left=4ex, "\Downarrow\mu_\alpha"] & SHS \arrow[r, "\delta^H S"] \arrow[dr, "S\alpha S"] \arrow[d, phantom, "\Downarrow S\alpha^*"] & HS^2 \arrow[dr, "\alpha S^2"] \arrow[d, phantom, "\Downarrow\alpha^* S"] & \\
SH \arrow[dr, "S\alpha"'] & S^2K \arrow[r, "S\delta^K"'] \arrow[d, "\mu K"'] \arrow[drr, phantom, "\Downarrow\overline{\mu}^K"] & SKS \arrow[r, "\delta^K S"'] & KS^2 \arrow[d, "K\mu"] \\
& SK \arrow[rr, "\delta^K"'] & & KS
\end{tikzcd}
=
\begin{tikzcd}
S^2H \arrow[r, "S\delta^H"] \arrow[d, "\mu H"'] \arrow[drr, phantom, "\Downarrow\overline{\mu}^H"] & SHS \arrow[r, "\delta^H S"] & HS^2 \arrow[d, "H\mu"'] \arrow[dr, "\alpha S^2"] \arrow[dr, phantom, "\Downarrow\alpha_\mu"] & \\
SH \arrow[rr, "\delta^H"'] \arrow[dr, "S\alpha"'] \arrow[drr, phantom, "\Downarrow\alpha^*"] & & HS \arrow[dr, "\alpha S"] & KS^2 \arrow[d, "K\mu"] \\
& SK \arrow[rr, "\delta^K"'] & & KS
\end{tikzcd}
\tag{α^*-1}$$

$$\begin{tikzcd}
& SH \arrow[dr, "\delta^H"] \arrow[d, phantom, "\Downarrow\overline{\eta}^H"] & \\
H \arrow[ur, "\eta H"] \arrow[rr, "H\eta"'] \arrow[d, "\alpha"'] \arrow[drr, phantom, "\Downarrow\alpha_\eta"] & \, & HS \arrow[d, "\alpha S"] \\
K \arrow[rr, "K\eta"'] & & KS
\end{tikzcd}
=
\begin{tikzcd}
& SH \arrow[d, "S\alpha"] \arrow[dr, "\delta^H"] & \\
H \arrow[ur, "\eta H"] \arrow[d, "\alpha"'] \arrow[r, phantom, "\Downarrow\eta_\alpha"] & SK \arrow[dr, "\delta^K"] \arrow[r, phantom, "\Downarrow\alpha^*"] \arrow[d, phantom, "\Downarrow\overline{\eta}^K"] & HS \arrow[d, "\alpha S"] \\
K \arrow[ur, "\eta K"] \arrow[rr, "K\eta"'] & \, & KS
\end{tikzcd}
\tag{α^*-2}$$

5.1.3 Pseudo-distributive laws and modifications

We extend our discussion further to modifications. Consider a modification $\zeta : \alpha \to \beta$ from H to K, with pseudo-distributive laws α^* and β^* with respect to δ^H and δ^K, as defined in the previous section. If ζ has a certain property, then it can be considered as an arrow from α^* to β^*, or, equivalently, in a sense, S distributes over ζ.

Definition 5.3. Let $\alpha, \beta : H \to K$ be pseudo-natural transformations with distributive laws α^* and β^* with respect to δ^H and δ^K. Given a modification $\zeta : \alpha \to \beta$, we say " S

distributes over ζ *with respect to* α^* *and* β^* ” if it satisfies the condition

$$
\begin{tikzcd}[column sep=large, row sep=large]
SH \arrow[r, "\delta^H"] \arrow[d, bend right=60, "S\beta"', ""{name=L}] \arrow[d, "S\alpha", ""'{name=R}] \arrow[dr, phantom, "\Downarrow\alpha^*"] & HS \arrow[d, "\alpha S"] \\
SK \arrow[r, "\delta^K"'] & KS
\arrow[Rightarrow, from=R, to=L, "S\zeta"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=large]
SH \arrow[r, "\delta^H"] \arrow[d, "S\beta"'] \arrow[dr, phantom, "\Downarrow\beta^*"] & HS \arrow[d, "\beta S"', ""{name=B}] \arrow[d, bend left=60, "\alpha S", ""'{name=A}] \\
SK \arrow[r, "\delta^K"'] & KS
\arrow[Rightarrow, from=A, to=B, "\zeta S"']
\end{tikzcd}
\tag{5.2}
$$

5.1.4 The 2-category Ps-Dist$^{\mathsf{S}}$

The data defined above form the 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ of pseudo-distributive laws of S over pseudo-endofunctors.

Proposition 5.4. *Given a pseudo-monad S on a 2-category $\mathbb{C}$, the following data form a 2-category we denote by $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$.*

- *the 0-cells of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ are pairs $(H,(\delta,\overline{\mu},\overline{\eta}))$ of a pseudo-endofunctor H on $\mathbb{C}$ and a pseudo-distributive law $(\delta : SH \to HS,\overline{\mu},\overline{\eta})$.*

- *A 1-cell in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ of type $\delta^H \to \delta^K$ is a pair (α,α^*) of a pseudo-natural transformation $\alpha : H \to K$ and a distributive law α^* of S over α with respect to δ^H and δ^K.*

- *A 2-cell ζ in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ of type $(\alpha,\alpha^*) \to (\beta,\beta^*)$ is a modification $\zeta : \alpha \to \beta$ that distributes over S with respect to α^* and β^*.*

Proof. The composition is given, for the 1-cells, by the composition of pseudo-natural transformations and the pasting of diagrams, and for 2-cells, by the usual horizontal composition. A routine calculation shows that $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ is well-defined. □

Notation 5.5. For a 0-cell $(H,(\delta^H,\overline{\mu},\overline{\eta}))$, we often suppress the pseudo-endofunctor and the 2-cells of the pseudo-distributive law and just write δ^H, and similarly, for the 1-cell (α,α^*) we often just write α^*, when it does not cause confusion.

We see later that the 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ is equivalent to the 2-category $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$ of liftings of pseudo-endofunctors to *Ps-S-Alg*, which we define in the next section.

5.2 The 2-category Lift$_{Ps\text{-}S\text{-}Alg}$

In this section we leave pseudo-distributive laws for a while and study the liftings of pseudo-endofunctors on $\mathbb{C}$ to *Ps-S-Alg*, the category of pseudo-S-algebras (Section 2.4). In Section 5.2.1 we give the definition of liftings of pseudo-endofunctors to *Ps-S-Alg*, followed by the definition of liftings of pseudo-natural transformations in Section 5.2.2 and the notion of lifting modifications in Section 5.2.3. Finally, we define the 2-category Lift$_{Ps\text{-}S\text{-}Alg}$ in Section 5.2.4.

5.2.1 Lifting of a pseudo-endofunctor H to *Ps-S-Alg*

Recall the definition of the forgetful 2-functor U from *Ps-S-Alg* to $\mathbb{C}$ (Section 2.4). A lifting of H to *Ps-S-Alg* is defined to be a pseudo-endofunctor $\widehat{H}$ on *Ps-S-Alg* that satisfies $U\widehat{H} = HU$, just as in the non-pseudo-case, except that the definition here includes the 2-cell part of the lifting in addition to those for objects and arrows, and the commutative diagrams are replaced by invertible 2-cells.

Definition 5.6. Given a pseudo-monad $S = (S, \mu, \eta, \tau, \lambda, \rho)$ and a pseudo-endofunctor H on a 2-category $\mathbb{C}$, a *lifting* of H to *Ps-S-Alg* is a pseudo-endofunctor $\widehat{H}$ on *Ps-S-Alg* for which $U\widehat{H} = HU$ holds, where U is the forgetful 2-functor from *Ps-S-Alg* to $\mathbb{C}$.

The above definition says that, given a 0-cell, a pseudo-S-algebra $\langle A, a, a_\mu, a_\eta\rangle$, 1-cell $(f, \overline{f}_{a,b}) : \langle A, a, a_\mu, a_\eta\rangle \to \langle B, b, b_\mu, b_\eta\rangle$, and a 2-cell $\chi : (f, \overline{f}_{a,b}) \to (g, \overline{g}_{a,b})$, in *Ps-S-Alg*, the lifting $\widehat{H}$ should satisfy

$$U\widehat{H}\langle A, a, a_\mu, a_\eta\rangle = HU\langle A, a, a_\mu, a_\eta\rangle = HA \tag{5.3a}$$

$$U\widehat{H}(f, \overline{f}_{a,b}) = HU(f, \overline{f}_{a,b}) = Hf \tag{5.3b}$$

$$U\widehat{H}\chi = HU\chi = H\chi \tag{5.3c}$$

Hence, for 0-cells, the equation (5.3a) states that $\widehat{H}$ sends a pseudo-S-algebra $\langle A, a, a_\mu, a_\eta\rangle$ to a pseudo-S-algebra of the following form:

$$\widehat{H}\langle A, a, a_\mu, a_\eta\rangle = \langle HA, \widehat{a}, \widehat{a}_\mu, \widehat{a}_\eta\rangle$$

where the structure map $\widehat{a}$ is of type $SHA \to HA$, and $\widehat{a}_\mu$ and $\widehat{a}_\eta$ are invertible 2-cells in $\mathbb{C}$ described in the following diagram:

$$\begin{tikzcd} S^2HA \arrow[r, "S\widehat{a}"] \arrow[d, "\mu_{HA}"'] \arrow[dr, phantom, "\Downarrow \widehat{a}_\mu"] & SHA \arrow[d, "\widehat{a}"] \\ SHA \arrow[r, "\widehat{a}"'] & HA \end{tikzcd} \qquad \begin{tikzcd} HA \arrow[r, "\eta_{HA}"] \arrow[dr, "\mathrm{id}_{HA}"'] & SHA \arrow[d, "\widehat{a}"] \arrow[dl, phantom, "\Downarrow \widehat{a}_\eta"] \\ & HA \end{tikzcd}$$

and they satisfy the coherence axioms for pseudo-algebras given in Section 2.4.

Notation 5.7. In the following discussion, whenever we need to distinguish more than one lifting, we use superscripts, as in $\widehat{H}\langle A, a, a_\mu, a_\eta\rangle = \langle A, \widehat{a}^H, \widehat{a}^H_\mu, \widehat{a}^H_\eta\rangle$ to indicate which lifting is associated to the hats.

For 1-cells, what (5.3b) means is that $\widehat{H}$ sends a pseudo-algebra map

$$(f, \overline{f}_{a,b}) : \langle A, a, a_\mu, a_\eta\rangle \to \langle B, b, b_\mu, b_\eta\rangle$$

to a pseudo-algebra map $(Hf, \widehat{f})$

$$\widehat{H}(f, \overline{f}_{a,b}) = (Hf, \overline{Hf}_{\widehat{a},\widehat{b}}) : \langle HA, \widehat{a}, \widehat{a}_\mu, \widehat{a}_\eta\rangle \to \langle HB, \widehat{b}, \widehat{b}_\mu, \widehat{b}_\eta\rangle,$$

where $\overline{Hf}_{\widehat{a},\widehat{b}}$ is an invertible 2-cell in $\mathbb{C}$ of the form:

$$\begin{tikzcd} SHA \arrow[r, "SHf"] \arrow[d, "\widehat{a}"'] \arrow[dr, phantom, "\Downarrow \overline{Hf}_{\widehat{a},\widehat{b}}"] & SHB \arrow[d, "\widehat{b}"] \\ HA \arrow[r, "Hf"'] & HB \end{tikzcd} \tag{5.4}$$

satisfying the two coherence axioms for pseudo-algebra maps given in Section 2.4.

Notation 5.8. In the following we sometimes write $\widehat{H}(\overline{f}_{a,b})$ instead of $\overline{Hf}_{\widehat{a},\widehat{b}}$ to denote this 2-cell.

For 2-cells, the equation (5.3c) states that $\widehat{H}$ sends an algebra 2-cell $\chi : (f, \overline{f}_{a,b}) \to (g, \overline{g}_{a,b})$ to an algebra 2-cell $H\chi : (Hf, \overline{Hf}_{\widehat{a},\widehat{b}}) \to (Hg, \overline{Hg}_{\widehat{a},\widehat{b}})$, satisfying the coherence axiom for algebra 2-cells.

5.2.2 Lifting pseudo-natural transformations

Given a pseudo-natural transformation $\alpha : H \to K : \mathbb{C} \to \mathbb{C}$, and liftings $\widehat{H}$, $\widehat{K}$ of H, K to *Ps-S-Alg*, we can consider a lifting of α.

Definition 5.9. A *lifting of* α *to Ps-S-Alg from* $\widehat{H}$ *to* $\widehat{K}$ is a pseudo-natural transformation $\widehat{\alpha} : \widehat{H} \to \widehat{K}$ for which $U\widehat{\alpha} = \alpha U$ holds.

From the equation $U\widehat{\alpha} = \alpha U$ we have the following two conditions:

1. for each pseudo-S-algebra $\langle A, a, a_\mu, a_\eta \rangle$, the component of $\widehat{\alpha}$ at this pseudo-algebra is given by the component α_A of α at A, that is, we have an invertible 2-cell $\overline{\alpha_{A,a}}$

$$\begin{tikzcd}
SHA \arrow[r, "S\alpha_A"] \arrow[d, "\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{A,a}}"] & SKA \arrow[d, "\widehat{a}^{\widehat{K}}"] \\
HA \arrow[r, "\alpha_A"'] & KA
\end{tikzcd}$$

   that makes α_A into a pseudo-algebra map from $\widehat{H}\langle A, a, a_\mu, a_\eta \rangle$ to $\widehat{K}\langle A, a, a_\mu, a_\eta \rangle$.

2. the pseudo-naturality of α extends to that of $\widehat{\alpha}$. Abbreviating pseudo-S-algebras $\langle A, a, a_\mu, a_\eta \rangle$ as $\langle A, a \rangle$, for each pseudo-algebra map $(f, \overline{f}_{a,b}) : \langle A, a \rangle \to \langle B, b \rangle$, the pseudo-naturality of $\widehat{\alpha}_{(f, \overline{f}_{a,b})}$ is defined to be an algebra 2-cell of the form

$$\begin{tikzcd}
\widehat{H}\langle A, a \rangle \arrow[r, "\alpha_A"] \arrow[d, "\widehat{H}f"'] \arrow[dr, phantom, "\Downarrow \widehat{\alpha}_{(f,\overline{f}_{a,b})}"] & \widehat{K}\langle A, a \rangle \arrow[d, "\widehat{K}f"] \\
\widehat{H}\langle B, b \rangle \arrow[r, "\alpha_B"'] & \widehat{K}\langle B, b \rangle
\end{tikzcd}$$

   that satisfies the pseudo-naturality conditions. In order for α to lift to $\widehat{\alpha}$, the component α_f of the pseudo-naturality of α at $f : A \to B$ should provide such an algebra 2-cell, hence it needs to satisfy the following additional condition:

$$
\begin{tikzcd}
SHA \arrow[r, "S\alpha_A"] \arrow[d, "\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{A,a}}"] & SKA \arrow[r, "SKf"] \arrow[d, "\widehat{a}^{\widehat{K}}"] \arrow[dr, phantom, "\Downarrow \widehat{K}(\overline{f}_{a,b})"] & SKB \arrow[d, "\widehat{b}^{\widehat{K}}"] \\
HA \arrow[r, "\alpha_A"] \arrow[dr, "Hf"'] & KA \arrow[r, "Kf"] \arrow[d, phantom, "\Downarrow \alpha_f"] & KB \\
& HB \arrow[ur, "\alpha_B"'] &
\end{tikzcd}
=
\begin{tikzcd}
& SKA \arrow[dr, "SKf"] \arrow[d, phantom, "\Downarrow S\alpha_f"] & \\
SHA \arrow[ur, "S\alpha_A"] \arrow[r, "SHf"] \arrow[d, "\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow \widehat{H}(\overline{f}_{a,b})"] & SHB \arrow[r, "S\alpha_B"] \arrow[d, "\widehat{b}^{\widehat{H}}"] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{B,b}}"] & SKB \arrow[d, "\widehat{b}^{\widehat{K}}"] \\
HA \arrow[r, "Hf"'] & HB \arrow[r, "\alpha_B"'] & KB
\end{tikzcd}
\tag{5.5}
$$

5.2.3 Lifting modifications

Given pseudo-natural transformations $\alpha, \beta : H \to K$ and their liftings $\widehat{\alpha}, \widehat{\beta} : \widehat{H} \to \widehat{K}$, a modification $\gamma : \alpha \to \beta$ with a certain property lifts uniquely to a modification from $\widehat{\alpha}$ to $\widehat{\beta}$.

Definition 5.10. We say " γ *lifts to Ps-S-Alg from* $\widehat{\alpha}$ *to* $\widehat{\beta}$ " if, for each pseudo-algebra $\langle A, a, a_\mu, a_\eta \rangle$, the component γ_A at A satisfies the condition required to be an algebra 2-cell from $(\alpha_A, \overline{\alpha_{A,a}})$ to $(\beta_A, \overline{\beta_{A,a}})$, i.e., it satisfies the equality

$$
\begin{tikzcd}
SHA \arrow[r, "S\alpha_A"] \arrow[d, "\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{A,a}}"] & SKA \arrow[d, "\widehat{a}^{\widehat{K}}"] \\
HA \arrow[r, shift left=2, "\alpha_A"] \arrow[r, shift right=2, "\beta_A"'] \arrow[r, phantom, "\Downarrow \gamma_A"] & KA
\end{tikzcd}
=
\begin{tikzcd}
SHA \arrow[r, shift left=2, "S\alpha_A"] \arrow[r, shift right=2, "S\beta_A"'] \arrow[r, phantom, "\Downarrow S\gamma_A"] \arrow[d, "\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow \overline{\beta_{A,a}}"] & SKA \arrow[d, "\widehat{a}^{\widehat{K}}"] \\
HA \arrow[r, "\beta_A"] & KA
\end{tikzcd}
\tag{5.6}
$$

5.2.4 The 2-category Lift$_{\textit{Ps-S-Alg}}$

All liftings of pseudo-endofunctors on $\mathbb{C}$ to *Ps-S-Alg* form a 2-category $\mathsf{Lift}_{\textit{Ps-S-Alg}}$, with pseudo-natural transformations and their liftings as 1-cells.

Proposition 5.11. *Let S be a pseudo-monad on $\mathbb{C}$. The following data form a 2-category we denote by $\mathsf{Lift}_{\textit{Ps-S-Alg}}$.*

- *The 0-cells of $\mathsf{Lift}_{\textit{Ps-S-Alg}}$ are pairs $(H, \widehat{H})$ of pseudo-endofunctors H on $\mathbb{C}$ and $\widehat{H}$ on Ps-S-Alg, such that $U\widehat{H} = HU$ holds.*

- *A 1-cell in* $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ *from* $(H,\widehat{H})$ *to* $(K,\widehat{K})$ *is a pair* $(\alpha,\widehat{\alpha})$ *of a pseudo-natural transformation* $\alpha : H \to K$ *and* $\widehat{\alpha} : \widehat{H} \to \widehat{K}$*, such that* $U\widehat{\alpha} = \alpha U$ *holds.*

- *A 2-cell in* $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ *from* $(\alpha,\widehat{\alpha})$ *to* $(\beta,\widehat{\beta})$ *is a modification* $\gamma : \alpha \to \beta$ *that lifts to* $Ps\text{-}S\text{-}Alg$ *from* $\widehat{\alpha}$ *to* $\widehat{\beta}$*.*

Proof. The composition of 1-cells is given by that of pseudo-natural transformations, and the rest follows by routine calculation. □

In the remainder of the chapter we establish the equivalence between $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ and $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$.

5.3 From liftings to pseudo-distributive laws

Our final goal in the following three sections is to construct 2-functors

$$\mathsf{Lift}_{Ps\text{-}S\text{-}Alg} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{Ps\text{-}Dist}^{\mathsf{S}}$$

and show that they form an equivalence of 2-categories. For the sake of clarity, we first give the construction for a fixed H; in this section we establish the construction from liftings to pseudo-distributive laws and in the next section in the opposite direction.

In Section 5.3.1 we start by studying some properties of a particular pseudo-S-algebra, whose structure map is a component μ_A of the multiplication μ of the monad S, which we will need for the rest of the section.

Then we move on to construct 0-cells of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ from those of $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ in Section 5.3.2 and similarly for 1-cells and 2-cells in Section 5.3.3 and Section 5.3.4. Using these constructions we define a 2-functor Φ from $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ to $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$.

5.3.1 Pseudo-*S*-algebra $\mu_{\mathbf{A}}$

Just like the non-pseudo case, the components of the multiplication μ of the pseudo-monad are always pseudo-algebra structure maps. In this section, we give a proof of this fact and then investigate some properties of such pseudo-algebras and their liftings.

Lemma 5.12. *For any 0-cell A of $\mathbb{C}$, the component μ_A of the multiplication μ is the structure map of a pseudo-S-algebra.*

Proof. We define $\mu_{A_\mu} = \tau_A$ and $\mu_{A_\eta} = \rho_A$ as shown in the diagrams below, and claim that $\langle SA, \mu_A, \tau_A, \rho_A \rangle$ constitutes a pseudo-S-algebra.

$$\begin{tikzcd} S^3A \arrow[r, "S\mu_A"] \arrow[d, "\mu_{SA}"'] \arrow[dr, phantom, "\Downarrow\tau_A"] & S^2A \arrow[d, "\mu_A"] \\ S^2A \arrow[r, "\mu_A"'] & SA \end{tikzcd} \qquad \begin{tikzcd} SA \arrow[r, "\eta_{SA}"] \arrow[dr, "id_{SA}"'] & S^2A \arrow[d, "\mu_A"] \arrow[dl, phantom, "\Downarrow\rho_A"] \\ & SA \end{tikzcd}$$

It follows immediately from the coherence axioms of pseudo-monads that these data satisfy the axioms for pseudo-S-algebras. □

Now we consider the value of $\widehat{H}$ at $\langle SA, \mu_A, \tau_A, \rho_A \rangle$. This pseudo-$S$-algebra is expressed as $\langle HSA, \widehat{\mu_A}, \widehat{\tau_A}, \widehat{\rho_A} \rangle$, with each component described as:

- a 1-cell (in $\mathbb{C}$) $\widehat{\mu_A} : SHSA \to HSA$
- invertible 2-cells $\widehat{\tau_A} : \widehat{\mu_A} \circ S\widehat{\mu_A} \to \widehat{\mu_A} \circ \mu_{HSA}$ and $\widehat{\rho_A} : \widehat{\mu_A} \circ \eta_{HSA} \to id_{HSA}$

$$\begin{tikzcd} S^2HSA \arrow[r, "S\widehat{\mu_A}"] \arrow[d, "\mu_{HSA}"'] \arrow[dr, phantom, "\Downarrow\widehat{\tau_A}"] & SHSA \arrow[d, "\widehat{\mu_A}"] \\ SHSA \arrow[r, "\widehat{\mu_A}"'] & HSA \end{tikzcd} \qquad \begin{tikzcd} HSA \arrow[r, "\eta_{HSA}"] \arrow[dr, "id_{HSA}"'] & SHSA \arrow[d, "\widehat{\mu_A}"] \arrow[dl, phantom, "\Downarrow\widehat{\rho_A}"] \\ & HSA \end{tikzcd}$$

satisfying the coherence axioms for pseudo-algebras.

Our next observation is that the three pieces of data described above are in a sense natural in A, in other words, the first and the other two are components of a pseudo-natural transformation and modifications, respectively.

Lemma 5.13. *The collections $\{\widehat{\mu_A}\}_A, \{\widehat{\tau_A}\}_A, \{\widehat{\rho_A}\}_A$ of the data of $\langle HSA, \widehat{\mu_A}, \widehat{\tau_A}, \widehat{\rho_A} \rangle$, for each A in $\mathbb{C}$, define a pseudo-natural transformation $\widehat{\mu} : SHS \to HS$, and invertible modifications $\widehat{\tau}$, $\widehat{\rho}$, respectively, in the 2-category $\mathbb{C}$.*

Proof. For the collection of the structure maps $\{\widehat{\mu}_A\}_A$, first note that for any 1-cell $f : A \to B$, Sf is a pseudo-algebra map between $\langle SA, \mu_A, \tau_A, \rho_A\rangle$ and $\langle SB, \mu_B, \tau_B, \rho_B\rangle$, and for any 2-cell $\chi : f \to g : A \to B$ in $\mathbb{C}$, $S\chi$ is an algebra 2-cell from Sf to Sg, both resulting from the fact μ being pseudo-natural transformation: the second component $\overline{Sf}_{\mu_A,\mu_B}$ of the pseudo-algebra map Sf is given by $(\mu_f^{A,B})^{-1}$, the component at f of the pseudo-naturality of μ. The justification for $S\chi$ being an algebra 2-cell follows from the fact that $\mu^{A,B}$ is a pseudo-natural transformation. The coherence axioms for Sf to be a pseudo-map are satisfied because of the fact that τ and ρ are suitable modifications.

Now consider the lifting of Sf. By letting $a = \mu_A$ and $b = \mu_B$ in the diagram (5.4) we obtain the following diagram for $\widehat{H}(\overline{Sf}_{\mu_A,\mu_B}) = \overline{HSf}_{\widehat{\mu}_A,\widehat{\mu}_B}$:

$$\begin{tikzcd}
SHSA \arrow[r, "SHSf"] \arrow[d, "\widehat{\mu}_A"'] \arrow[dr, phantom, "\Downarrow \overline{HSf}_{\widehat{\mu}_A,\widehat{\mu}_B}"] & SHSB \arrow[d, "\widehat{\mu}_B"] \\
HSA \arrow[r, "HSf"'] & HSB
\end{tikzcd} \tag{5.7}$$

We define pseudo-naturality $\widehat{\mu}_f^{A,B}$ to be $(\overline{HSf}_{\widehat{\mu}_A,\widehat{\mu}_B})^{-1}$. This $\widehat{\mu}^{A,B}$ defined in this way is indeed a pseudo-natural transformation, because the conditions for an algebra 2-cell $\widehat{H}S\chi = HS\chi$ provide the necessary pseudo-naturality.

For $\{\widehat{\tau}_A\}_A$, we show that this defines an invertible modification: a component $\widehat{\tau}_A$ is by definition invertible and of type $\widehat{\mu}_A \circ S\widehat{\mu}_A \to \widehat{\mu}_A \circ \mu_{HSA}$, whose domain and codomain we now know are pseudo-natural transformations. Now, given a 1-cell $f : A \to B$, the axiom for modifications are satisfied as follows: abbreviating $(\overline{HSf}_{\widehat{\mu}_A,\widehat{\mu}_B})^{-1}$ as $\overline{HSf}^{-1}$,

$$\begin{tikzcd}
S^2HSA \arrow[r, "S\widehat{\mu}_A"] \arrow[d, "S^2HSf"'] \arrow[dr, phantom, "\Downarrow S\overline{HSf}^{-1}"] & SHSA \arrow[r, "\widehat{\mu}_A"] \arrow[d, "SHSf"] \arrow[dr, phantom, "\Downarrow \overline{HSf}^{-1}"] & HSA \arrow[d, "HSf"] \\
S^2HSB \arrow[r, "S\widehat{\mu}_B"] \arrow[dr, "\mu_{HSB}"'] & SHSB \arrow[r, "\widehat{\mu}_B"] \arrow[d, phantom, "\Downarrow \widehat{\tau}_B"] & HSB \\
 & SHSB \arrow[ur, "\widehat{\mu}_B"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & SHSA \arrow[dr, "\widehat{\mu}_A"] \arrow[d, phantom, "\Downarrow \widehat{\tau}_A"] & \\
S^2HSA \arrow[ur, "S\widehat{\mu}_A"] \arrow[r, "\mu_{HSA}"] \arrow[d, "S^2HSf"'] \arrow[dr, phantom, "\Downarrow \mu_{HSf}"] & SHSA \arrow[r, "\widehat{\mu}_A"] \arrow[d, "SHSf"] \arrow[dr, phantom, "\Downarrow \overline{HSf}^{-1}"] & HSA \arrow[d, "HSf"] \\
S^2HSB \arrow[r, "\mu_{HSB}"'] & SHSB \arrow[r, "\widehat{\mu}_B"'] & HSB
\end{tikzcd}$$

the equality holds because by suitably rearranging the invertible 2-cells one can obtain the diagram for the associativity axiom for $\widehat{Sf}$ to be a pseudo-algebra map. Hence $\widehat{\tau}$ is an invertible modification. A similar argument also holds for $\widehat{\rho} : \widehat{\mu} \circ \eta HS \to idHS$. The axiom for modifications it satisfies is given below.

$$
\begin{tikzcd}
 & SHSA \arrow[d, "SHSf"] \arrow[dr, "\widehat{\mu}_A"] & & & & SHSA \arrow[dr, "\widehat{\mu}_A"] \arrow[d, phantom, "\Downarrow\widehat{\rho}_A"] & \\
HSA \arrow[ur, "\eta_{HSA}"] \arrow[r, phantom, "\Downarrow\eta_{HSf}"] \arrow[d, "HSf"'] & SHSB \arrow[r, phantom, "\Downarrow\overline{HSf}^{-1}"] \arrow[dr, "\widehat{\mu}_B"] \arrow[d, phantom, "\Downarrow\widehat{\rho}_B"] & HSA \arrow[d, "HSf"] & = & HSA \arrow[ur, "\eta_{HSA}"] \arrow[rr, "id_{HSA}"'] \arrow[d, "HSf"'] & & HSA \arrow[d, "HSf"] \\
HSB \arrow[ur, "\eta_{HSB}"] \arrow[rr, "id_{HSB}"'] & & HSB & & HSB \arrow[rr, "id_{HSB}"'] & & HSB
\end{tikzcd}
$$

□

To end the section, we state the fact that any structure map a is a pseudo-algebra map from the algebra μ_A to a:

Lemma 5.14. *For any pseudo-S-algebra $\langle A, a, a_\mu, a_\eta \rangle$, the structure map a is a pseudo-S-algebra map from $\langle SA, \mu_A, \tau_A, \rho_A \rangle$ to $\langle A, a, a_\mu, a_\eta \rangle$.*

Proof. Let the 2-cell $\overline{a}_{a,\mu_A}$ for the pseudo-algebra map a to be a_μ. Then it is immediate from the definitions that it satisfies the necessary axioms. □

Now consider the value of a lifting at such a pseudo-algebra map. The lifting of this map (a, a_μ) gives a pseudo-S-algebra map

$$(Ha, \widehat{H}(\overline{a}_{a,\mu_A})) : \langle HSA, \widehat{\mu}_A, \widehat{\tau}_A, \widehat{\rho}_A \rangle \to \langle HA, \widehat{a}, \widehat{a}_\mu, \widehat{a}_\eta \rangle$$

where the 2-cell $\widehat{H}(\overline{a}_{a,\mu_A})$ is of the form

$$
\begin{tikzcd}
SHSA \arrow[r, "SHa"] \arrow[d, "\widehat{\mu}_A"'] \arrow[dr, phantom, "\Downarrow\widehat{H}(\overline{a}_{a,\mu_A})"] & HSA \arrow[d, "\widehat{a}"] \\
HSA \arrow[r, "Ha"'] & HA
\end{tikzcd}
$$

satisfying the axioms for pseudo-algebra maps.

5.3.2 0-cells

We show the construction of a pseudo-distributive law from a lifting in the proof of the following proposition. The pseudo-natural transformation of the pseudo-S-algebras $\widehat{\mu}$ plays a central role in the construction.

Proposition 5.15. *Given a pseudo-monad $(S, \mu, \eta, \tau, \lambda, \rho)$ and a pseudo-endofunctor H on a 2-category $\mathbb{C}$, a lifting of H to Ps-S-Alg gives rise to a pseudo-distributive law of S over H.*

Proof. In order to prove the proposition we construct a function Φ^H from the set of all liftings of H to the set of all pseudo-distributive laws of S over H as follows. (We omit the superscript H in the rest of this section.) Given a lifting $\widehat{H}$, the value

$$\Phi(\widehat{H}) = (\Phi(\widehat{H}), \overline{\mu}^{\Phi(\widehat{H})}, \overline{\eta}^{\Phi(\widehat{H})})$$

is given by first defining $\Phi(\widehat{H}) = \widehat{\mu}^{\widehat{H}} \circ SH\eta$, that is,

$$\Phi(\widehat{H}) : SH \xrightarrow{SH\eta} SHS \xrightarrow{\widehat{\mu}^{\widehat{H}}} HS, \tag{5.8}$$

using the pseudo-natural transformation $\widehat{\mu}^{\widehat{H}}$ discussed in Lemma 5.13. This is easily seen to be a pseudo-natural transformation.

Next we define the components of the invertible modifications: $\overline{\mu}_A^{\Phi(\widehat{H})}$ is defined to be the invertible 2-cell described in the following diagrams:

$$\begin{tikzcd}
S^2HA \arrow[r, "S^2H\eta_A"] \arrow[dd, "\mu_{HA}"'] & S^2HSA \arrow[r, "S\widehat{\mu}_A^{\widehat{H}}"] \arrow[ddr, "\mu_{HSA}"] & SHSA \arrow[r, "SH\eta_{SA}"] \arrow[dr, "id"'] & SHS^2A \arrow[r, "\widehat{\mu}_{SA}^{\widehat{H}}"] \arrow[d, "SH\mu_A"] \arrow[dl, phantom, "\Downarrow SH\rho_A"] & HS^2A \arrow[dd, "H\mu_A"] \\
\arrow[r, phantom, "\Downarrow \mu^{-1}_{H\eta_A}"] & {} & \arrow[r, phantom, "\Downarrow \widehat{\tau}_A^{\widehat{H}}"] & SHSA \arrow[dr, "\widehat{\mu}_A^{\widehat{H}}"] \arrow[r, phantom, "\Downarrow \widehat{H}(\overline{\mu_A}_{\mu_A, \mu_{SA}})^{-1}"] & {} \\
SHA \arrow[rr, "SH\eta_A"'] & & SHSA \arrow[rr, "\widehat{\mu}_A^{\widehat{H}}"'] & & HSA
\end{tikzcd} \tag{5.9a}$$

and $\overline{\eta}_A^{\Phi(\widehat{H})}$:

$$
\begin{tikzcd}
HA \arrow[rr, "\eta_{HA}"] \arrow[dr, "H\eta_A"'] & & SHA \arrow[d, "SH\eta_A"] \\
& HS \arrow[ur, phantom, "\Downarrow \eta_{H\eta_A}"] \arrow[r, "\eta_{HSA}"] \arrow[dr, "id"'] & SHSA \arrow[d, "\widehat{\mu}_A^{\widehat{H}}"] \arrow[dl, phantom, "\Downarrow \widehat{\rho}_A^{\widehat{H}}"] \\
& & HSA
\end{tikzcd}
\tag{5.9b}
$$

These modifications satisfy the axioms for the pseudo-distributive laws (H-1)–(H-3). Hence Φ^H indeed defines a function as required, proving the proposition. ☐

5.3.3 1-cells

Now we have a look at 1-cells of $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, i.e., pseudo-natural transformations that lift to *Ps-S-Alg*. The following proposition gives the construction of pseudo-distributive laws over α from a lifting of α.

Proposition 5.16. *A lifting $\widehat{\alpha} : \widehat{H} \to \widehat{K}$ of $\alpha : H \to K$ to Ps-S-Alg induces a pseudo-distributive law $\Phi(\widehat{\alpha})$ with respect to the induced pseudo-distributive laws $\Phi^H(\widehat{H})$ and $\Phi^K(\widehat{K})$.*

Proof. Recall that the induced pseudo-distributive laws are given by $\Phi^H(\widehat{H}) = \widehat{\mu}^{\widehat{H}} \circ SH\eta$ and $\Phi^K(\widehat{K}) = \widehat{\mu}^{\widehat{K}} \circ SK\eta$. We define the component $\Phi(\widehat{\alpha})_A$ of the modification $\Phi(\widehat{\alpha})$ at A as

$$
\begin{tikzcd}
SHA \arrow[r, "SH\eta_A"] \arrow[d, "S\alpha_A"'] & SHSA \arrow[r, "\widehat{\mu}_A^{\widehat{H}}"] \arrow[d, "S\alpha_{SA}"] \arrow[dl, phantom, "\Downarrow S\alpha_{\eta_A}^{-1}"] & HSA \arrow[d, "\alpha_{SA}"] \arrow[dl, phantom, "\Downarrow \overline{\alpha_{SA,\mu_A}}^{-1}"] \\
SKA \arrow[r, "SK\eta_A"'] & SKSA \arrow[r, "\widehat{\mu}_A^{\widehat{K}}"'] & KSA
\end{tikzcd}
$$

We need to verify that this satisfies the axiom for modifications and (α^*-1) and (α^*-2). First, the axiom for modifications holds as shown below by the pseudo-naturality of α

and by the axiom for algebra 2-cell α_{Sf}:

$$
\begin{tikzcd}
 & SHA \arrow[r, "SH\eta_A"] \arrow[dl, "SHf"'] \arrow[d, "S\alpha_A"] \arrow[dr, phantom, "\Downarrow S\alpha^{-1}_{\eta_A}"] & SHSA \arrow[r, "\widehat{\mu}^{\widehat{H}}_A"] \arrow[d, "S\alpha_{SA}"] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{SA,\mu_A}}^{-1}"] & HSA \arrow[d, "\alpha_{SA}"] \\
SHB \arrow[d, "S\alpha_B"'] \arrow[r, phantom, "\Leftarrow S\alpha_f"] & SKA \arrow[r, "SK\eta_A"'] \arrow[dl, "SKf"] \arrow[d, phantom, "\Leftarrow SK\eta_f"] & SKSA \arrow[r, "\widehat{\mu}^{\widehat{K}}_A"'] \arrow[dl, "SKSf"] \arrow[d, phantom, "\Leftarrow \widehat{\mu}^{\widehat{K}}_f"] & KSA \arrow[dl, "KSf"] \\
SKB \arrow[r, "SK\eta_B"'] & SKSB \arrow[r, "\widehat{\mu}^{\widehat{K}}_B"'] & KSB &
\end{tikzcd}
=
\begin{tikzcd}
 & SHA \arrow[r, "SH\eta_A"] \arrow[dl, "SHf"'] \arrow[d, phantom, "\Downarrow SH\eta_f"] & SHSA \arrow[r, "\widehat{\mu}^{\widehat{H}}_A"] \arrow[dl, "SHSf"] \arrow[d, phantom, "\Downarrow \widehat{\mu}^{\widehat{H}}_f"] & HSA \arrow[dl, "HSf"] \arrow[d, "\alpha_{SA}"] \\
SHB \arrow[r, "SH\eta_B"'] \arrow[d, "S\alpha_B"'] \arrow[dr, phantom, "\Downarrow S\alpha^{-1}_{\eta_B}"] & SHSB \arrow[r, "\widehat{\mu}^{\widehat{H}}_B"'] \arrow[d, "S\alpha_{SB}"] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{SB,\mu_B}}^{-1}"] & HSB \arrow[d, "\alpha_{SB}"] \arrow[r, phantom, "\Leftarrow \alpha_{Sf}"] & KSA \arrow[dl, "KSf"] \\
SKB \arrow[r, "SK\eta_B"'] & SKSB \arrow[r, "\widehat{\mu}^{\widehat{K}}_B"'] & KSB &
\end{tikzcd}
$$

Next, the axiom (α^*-1):

$$
\begin{tikzcd}
S^2HA \arrow[r, "S^2H\eta_A"] \arrow[dr, "S^2\alpha_A"'] \arrow[dd, "\mu_{HA}"'] & S^2HSA \arrow[r, "S\widehat{\mu}^{\widehat{H}}_A"] \arrow[dr, "S^2\alpha_{SA}"] \arrow[d, phantom, "\Downarrow S^2\alpha^{-1}_{\eta_A}"] & SHSA \arrow[r, "SH\eta_{SA}"] \arrow[dr, "S\alpha_{SA}"] \arrow[d, phantom, "\Downarrow S\overline{\alpha_{SA,\mu_A}}^{-1}"] & SHS^2A \arrow[r, "\widehat{\mu}^{\widehat{H}}_{SA}"] \arrow[dr, "S\alpha_{S^2A}"] \arrow[d, phantom, "\Downarrow S\alpha^{-1}_{\eta_{SA}}"] & HS^2A \arrow[dr, "\alpha_{S^2A}"] \arrow[d, phantom, "\Downarrow \overline{\alpha_{S^2A,\mu_{SA}}}^{-1}"] & \\
 & S^2KA \arrow[r, "S^2K\eta_A"] \arrow[dd, "\mu_{KA}"'] \arrow[l, phantom, "\Downarrow \mu^{-1}_{\alpha_A}"] \arrow[ddr, phantom, "\Downarrow \mu^{-1}_{K\eta_A}"] & S^2KSA \arrow[r, "S\widehat{\mu}^{\widehat{K}}_A"] \arrow[ddr, "\mu_{KSA}"] & SKSA \arrow[r, "SK\eta_{SA}"] \arrow[dr, "id"'] \arrow[d, phantom, "\Downarrow \widehat{\tau}^{\widehat{K}}_A"] \arrow[r, phantom, shift right=3, "\Downarrow SK\rho_A"] & SKS^2A \arrow[r, "\widehat{\mu}^{\widehat{K}}_{SA}"] \arrow[d, "SK\mu_A"] \arrow[dr, phantom, "\Downarrow \widehat{K}(\overline{\mu_A}_{\mu_A,\mu_{SA}})^{-1}"] & KS^2A \arrow[dd, "K\mu_A"] \\
SHA \arrow[dr, "S\alpha_A"'] & & & & SKSA \arrow[dr, "\widehat{\mu}^{\widehat{K}}_A"] & \\
 & SKA \arrow[rr, "SK\eta_A"'] & & SKSA \arrow[rr, "\widehat{\mu}^{\widehat{K}}_A"'] & & KSA
\end{tikzcd}
$$

equals

$$
\begin{tikzcd}
S^2HA \arrow[r, "S^2H\eta_A"] \arrow[dd, "\mu_{HA}"'] \arrow[ddr, phantom, "\Downarrow \mu^{-1}_{H\eta_A}"] & S^2HSA \arrow[r, "S\widehat{\mu}^{\widehat{H}}_A"] \arrow[ddr, "\mu_{HSA}"] & SHSA \arrow[r, "SH\eta_{SA}"] \arrow[dr, "id"'] \arrow[dd, phantom, "\Downarrow \widehat{\tau}^{\widehat{H}}_A"] \arrow[r, phantom, shift right=3, "\Downarrow SH\rho_A"] & SHS^2A \arrow[r, "\widehat{\mu}^{\widehat{H}}_{SA}"] \arrow[d, "SH\mu_A"] \arrow[dr, phantom, "\Downarrow \widehat{H}(\overline{\mu_A}_{\mu_A,\mu_{SA}})^{-1}"] & HS^2A \arrow[dr, "\alpha_{S^2A}"] \arrow[dd, "H\mu_A"] & \\
 & & & SHSA \arrow[dr, "\widehat{\mu}_A"] & \arrow[r, phantom, "\Downarrow \alpha_{\mu_A}"] & KS^2A \arrow[dd, "K\mu_A"] \\
SHA \arrow[rr, "SH\eta_A"] \arrow[dr, "S\alpha_A"'] \arrow[drr, phantom, "\Downarrow S\alpha^{-1}_{\eta_A}"] & & SHSA \arrow[rr, "\widehat{\mu}^{\widehat{H}}_A"] \arrow[dr, "S\alpha_{SA}"] \arrow[drr, phantom, "\Downarrow \overline{\alpha_{SA,\mu_A}}^{-1}"] & & HSA \arrow[dr, "\alpha_{SA}"] & \\
 & SKA \arrow[rr, "SK\eta_A"'] & & SKSA \arrow[rr, "\widehat{\mu}^{\widehat{K}}_A"'] & & KSA
\end{tikzcd}
$$

The proof is, in the order of application, by the pseudo-naturality of μ, by that of α, by the axiom for the algebra 2-cell α_{μ_A}, and by the axiom for the pseudo-algebra map $\overline{\alpha_{SA,\mu_A}}$.

Finally for the axiom (α^*-2), the following equality holds by the axiom for the pseudo-algebra map $\overline{\alpha_{SA,\mu_A}}$, and by the pseudo-naturality of η.

$$
\begin{tikzcd}
& HA \arrow[ddl, "\eta_{HA}"'] \arrow[dr, "H\eta"] \arrow[dd, phantom, "\Downarrow \eta_{H\eta_A}"] & \\
& & HSA \arrow[dl, "\eta_{HSA}"'] \arrow[d, "id"] \arrow[dl, phantom, "\Downarrow(\widehat{\rho}^{\widehat{H}}_A)^{-1}", near end] \\
SHA \arrow[r, "SH\eta_A"] \arrow[d, "S\alpha_A"'] \arrow[dr, phantom, "\Downarrow S\alpha^{-1}_{\eta_A}"] & SHSA \arrow[r, "\widehat{\mu}^{\widehat{H}}_A"'] \arrow[d, "S\alpha_{SA}"] \arrow[dr, phantom, "\Downarrow \overline{\alpha_{SA,\mu_A}}^{-1}"] & HSA \arrow[d, "\alpha_{SA}"] \\
SKA \arrow[r, "SK\eta_A"'] & SKSA \arrow[r, "\widehat{\mu}^{\widehat{K}}_A"'] & KSA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& HA \arrow[d, "\alpha_A"] \arrow[ddl, "\eta_{HA}"'] \arrow[ddrr, "H\eta_A"] & & \\
& KA \arrow[ddl, "\eta_{KA}"'] \arrow[dr, "K\eta_A"] \arrow[l, phantom, "\Downarrow \eta_{\alpha_A}"] \arrow[r, phantom, "\Downarrow \alpha_{\eta_A}"] \arrow[d, phantom, "\Downarrow \eta_{K\eta_A}"] & & \\
SHA \arrow[d, "S\alpha_A"'] & & KSA \arrow[dl, "\eta_{KSA}"'] \arrow[dr, "id"] \arrow[d, phantom, "\Downarrow(\widehat{\rho}^{\widehat{K}}_A)^{-1}"] & HSA \arrow[d, "\alpha_{SA}"] \\
SKA \arrow[r, "SK\eta_A"'] & SKSA \arrow[rr, "\widehat{\mu}^{\widehat{K}}_A"'] & & KSA
\end{tikzcd}
$$

This concludes the proof of the proposition. □

5.3.4 2-cells

Proposition 5.17. *For any modification $\gamma : \alpha \to \beta : H \to K$, where H and K are pseudo-endofunctors on $\mathbb{C}$, the following holds: if γ is a 2-cell in $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ and $\gamma : \widehat{\alpha} \to \widehat{\beta} : \widehat{H} \to \widehat{K}$, then γ is a 2-cell in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\gamma : \Phi(\widehat{\alpha}) \to \Phi(\widehat{\beta})$.*

Proof. The assumption requires γ to satisfy the axiom (5.6). Then the condition (5.2) that γ needs to satisfy to be a 2-cell $\gamma : \Phi(\widehat{\alpha}) \to \Phi(\widehat{\beta})$ in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ follows from the axiom (5.6) and that for modifications. □

Now we are ready to define the 2-functor Φ. For 0-cells, define the function Φ_0 using Φ^H for each pseudo-endofunctors H on $\mathbb{C}$, defined in Proposition 5.15, i.e., we define $\Phi_0(H, \widehat{H}) = (H, \Phi^H(\widehat{H}))$, where $\Phi^H(\widehat{H}) = \widehat{\mu}^{\widehat{H}} \circ SH\eta$. For 1-cells, given a 1-cell $(\alpha, \widehat{\alpha})$ in $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, we define $\Phi(\alpha, \widehat{\alpha}) = (\alpha, \Phi(\widehat{\alpha}))$ as in Proposition 5.16. And, for any 2-cell γ in $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, Lemma 5.17 justifies defining $\Phi(\gamma) = \gamma$.

Proposition 5.18. Φ *defined as above is a 2-functor.*

Proof. Follows from routine calculation. □

5.4 From pseudo-distributive laws to liftings

In this section, we show the construction in the opposite direction, from pseudo-distributive laws to liftings. The structure of this section is similar to that of the previous section. First we construct 0-cells of $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$ from those of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ in Section 5.4.1 and then similarly for 1-cells in Section 5.4.2 and 2-cells in Section 5.4.3.

5.4.1 0-cells

Proposition 5.19. *Given a pseudo-monad $(S,\mu,\eta,\tau,\lambda,\rho)$ and a pseudo-endofunctor H on a 2-category $\mathbb{C}$, a pseudo-distributive law of S over H gives rise to a lifting of H to Ps-S-Alg.*

Proof. We construct a function $\Psi^H$from the set of all pseudo-distributive laws of S over H to the set of all liftings of H to *Ps-S-Alg*. (Again, we omit the superscript H in the rest of this section.) Given a pseudo-distributive law $(\delta : SH \to HS, \overline{\mu}, \overline{\eta})$, we define the value $\Psi(\delta)$ at this pseudo-distributive law to be the pseudo-endofunctor on *Ps-S-Alg* that sends a pseudo-S-algebra $\langle A, a, a_\mu, a_\eta \rangle$, to

$$\Psi(\delta)\langle A, a, a_\mu, a_\eta \rangle = \langle HA, \widehat{a}^{\Psi(\delta)}, \widehat{a}_\mu^{\Psi(\delta)}, \widehat{a}_\eta^{\Psi(\delta)} \rangle$$

where the structure map is defined to be $\widehat{a}^{\Psi(\delta)} = Ha \circ \delta_A$, and the invertible 2-cells $\widehat{a}_\mu^{\Psi(\delta)}$ and $\widehat{a}_\eta^{\Psi(\delta)}$ are defined as described in the following diagrams:

$$\begin{tikzcd}
S^2HA \arrow[r, "S\delta_A"] \arrow[dd, "\mu_{HA}"'] & SHSA \arrow[r, "SHa"] \arrow[d, "\delta_{SA}"'] \arrow[dr, phantom, "\Downarrow \delta_a^{-1}"] & SHA \arrow[d, "\delta_A"] \\
\arrow[r, phantom, "\Downarrow \overline{\mu}_A"] & HS^2A \arrow[r, "HSa"'] \arrow[d, "H\mu_A"'] \arrow[dr, phantom, "\Downarrow Ha_\mu"] & HSA \arrow[d, "Ha"] \\
SHA \arrow[r, "\delta_A"'] & HSA \arrow[r, "Ha"'] & HA
\end{tikzcd} \tag{5.10a}$$

$$\begin{tikzcd}
HA \arrow[r, "\eta_{HA}"] \arrow[dr, "H\eta_A" description] \arrow[ddr, "id"'] \arrow[dr, phantom, shift left=4ex, "\Downarrow \overline{\eta}_A"] \arrow[ddr, phantom, shift left=3ex, "\Downarrow Ha_\eta"] & SHA \arrow[d, "\delta_A"] \\
& HSA \arrow[d, "Ha"] \\
& HA
\end{tikzcd} \tag{5.10b}$$

One can show that data defined as above satisfy the necessary coherece conditions for pseudo-algebras. Hence $\langle HA, Ha\circ\delta_A, \widehat{a}_\mu^{\Psi(\delta)}, \widehat{a}_\eta^{\Psi(\delta)}\rangle$ is indeed a pseudo-S-algebra. For the algebra maps, given a pseudo-S-algebra map $(f,\overline{f}_{a,b}) : \langle A, a, a_\mu, a_\eta\rangle \to \langle B, b, b_\mu, b_\eta\rangle$, we define $\Psi(\delta)(f,\overline{f}_{a,b})$ to be $(Hf, \Psi(\delta)(\overline{f}_{a,b}))$, where the invertible 2-cell $\Psi(\delta)(\overline{f}_{a,b})$ is given as

$$\begin{tikzcd}
SHA \arrow[r, "\delta_A"] \arrow[d, "SHf"'] \arrow[dr, phantom, "\Uparrow \delta_f^{-1}"] & HSA \arrow[r, "Ha"] \arrow[d, "HSf" description] \arrow[dr, phantom, "\Uparrow H\overline{f}_{a,b}"] & HA \arrow[d, "Hf"] \\
SHB \arrow[r, "\delta_B"'] & HSB \arrow[r, "Hb"'] & HB
\end{tikzcd}$$

It is routine to verify that this 2-cell satisfies the axioms for pseudo-algebra maps and that it indeed is a pseudo-S-algebra map from $\langle HA, Ha\circ\delta_A, \widehat{a}_\mu^{\Psi(\delta)}, \widehat{a}_\eta^{\Psi(\delta)}\rangle$ to $\langle HB, Hb\circ\delta_B, \widehat{b}_\mu^{\Psi(\delta)}, \widehat{b}_\eta^{\Psi(\delta)}\rangle$. For algebra 2-cells, given $\chi : (f,\overline{f}_{a,b}) \to (g,\overline{g}_{a,b})$, we define $\Psi(\delta)(\chi) = H\chi$. Again, it is easy to see that this is well-defined. This proves that Ψ^H is well defined. Since it is obviously a lifting of H, this completes the proof of the proposition.

□

5.4.2 1-cells

Proposition 5.20. *Let $\delta^H : SH \to HS$ and $\delta^K : SK \to KS$ be pseudo-distributive laws over pseudo-endofucntors H and K, and $\alpha : H \to K$ be a pseudo-natural transformation. Then a pseudo-distributive law α^* of S over α with respect to δ^H and δ^K induces a lifting $\Psi(\alpha^*)$ of α to Ps-S-Alg from $\Psi^H(\delta^H)$ to $\Psi^K(\delta^K)$.*

Proof. As we have seen in Section 5.2.2, to define a lifting of α we need (1) to construct, for each pseudo-S-algebra $\langle A, a, a_\mu, a_\eta\rangle$, a 2-cell, which we name $\overline{\Psi(\alpha^*)_{A,a}}$, that makes α_A a pseudo-algebra map from $\Psi^H(\delta^H)\langle A, a, a_\mu, a_\eta\rangle$ to $\Psi^K(\delta^K)\langle A, a, a_\mu, a_\eta\rangle$, and (2) to show that the construction in (1) makes the pseudo-naturality of α also lift.

For (1), we define the 2-cell $\overline{\Psi(\alpha^*)_{A,a}}$ as follows:

$$\begin{tikzcd}
SHA \arrow[r, "S\alpha_A"] \arrow[d, "\delta^H_A"'] \arrow[dr, phantom, "\Downarrow \alpha^{*\,-1}_A"] & SKA \arrow[d, "\delta^K_A"] \\
HSA \arrow[r, "\alpha_{SA}"] \arrow[d, "Ha"'] \arrow[dr, phantom, "\Downarrow \alpha_a"] & KSA \arrow[d, "Ka"] \\
HA \arrow[r, "\alpha_A"'] & KA
\end{tikzcd}$$

We need to verify that this defines a pseudo-algebra map. For the axiom with a_μ, the following equality holds by, in the order of application, the modification axiom for α^*, by the pseudo-naturality of α and by the axiom (α^*-1):

$$\begin{tikzcd}
 & S^2KA \arrow[r, "S\delta^K_A"] & SKSA \arrow[r, "SKa"] & SKA \arrow[d, "\delta^K_A"] \\
S^2HA \arrow[ur, "S^2\alpha_A"] \arrow[r, "S\delta^H_A"] \arrow[dd, "\mu_{HA}"'] \arrow[u, phantom, "\Downarrow S(\alpha^*_A)^{-1}"] & SHSA \arrow[ur, "S\alpha_{SA}"] \arrow[r, "SHa"] \arrow[d, "\delta^H_{SA}"] \arrow[u, phantom, "\Downarrow S\alpha_a"] & SHA \arrow[ur, "S\alpha_A"] \arrow[d, "\delta^H_A"] \arrow[r, phantom, "\Downarrow \alpha^{*\,-1}_A"] & KSA \arrow[d, "Ka"] \\
\arrow[r, phantom, "\Downarrow \overline{\mu}^H_A"] & HS^2A \arrow[r, "HSa"'] \arrow[d, "H\mu_A"] \arrow[u, phantom, "\Downarrow (\delta^H_a)^{-1}"] \arrow[dr, phantom, "\Downarrow Ha_\mu"] & HSA \arrow[ur, "\alpha_{SA}"] \arrow[d, "Ha"] \arrow[r, phantom, "\Downarrow \alpha_a"] & KA \\
SHA \arrow[r, "\delta^H_A"'] & HSA \arrow[r, "Ha"'] & HA \arrow[ur, "\alpha_A"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & S^2KA \arrow[r, "S\delta^K_A"] \arrow[dd, "\mu_{KA}"] & SKSA \arrow[r, "SKa"] \arrow[d, "\delta^K_{SA}"'] \arrow[dr, phantom, "\Downarrow (\delta^K_a)^{-1}"] & SKA \arrow[d, "\delta^K_A"] \\
S^2HA \arrow[ur, "S^2\alpha_A"] \arrow[dd, "\mu_{HA}"'] \arrow[dr, phantom, "\Downarrow \mu^{-1}_{\alpha_A}"] & \arrow[r, phantom, "\Downarrow \overline{\mu}^K_A"] & KS^2A \arrow[r, "KSa"'] \arrow[d, "K\mu_A"'] \arrow[dr, phantom, "\Downarrow Ka_\mu"] & KSA \arrow[d, "Ka"] \\
 & SKA \arrow[r, "\delta^K_A"'] \arrow[d, phantom, "\Downarrow \alpha^{*\,-1}_A"] & KSA \arrow[r, "Ka"'] \arrow[d, phantom, "\Downarrow \alpha_a"] & KA \\
SHA \arrow[ur, "S\alpha_A"] \arrow[r, "\delta^H_A"'] & HSA \arrow[ur, "\alpha_{SA}"] \arrow[r, "Ha"'] & HA \arrow[ur, "\alpha_A"'] &
\end{tikzcd}$$

and for the axiom with a_η, we have the following equality by the axiom (α^*-2) and by the pseudo-naturality of α:

$$
\begin{tikzcd}
 & KA \arrow[drr, "\eta_{KA}"] \arrow[d, phantom, "\Downarrow \eta^{-1}_{\alpha_A}"] & & \\
HA \arrow[ur, "\alpha_A"] \arrow[rr, "\eta_{HA}"] \arrow[drr, "H\eta_A"'] \arrow[ddrr, "id"'] & \, \arrow[d, phantom, "\Downarrow \overline{\eta}^H_A"] & SHA \arrow[r, "S\alpha_A"] \arrow[d, "\delta^H_A"'] \arrow[dr, phantom, "\Downarrow \alpha^{*\,-1}_A"] & SKA \arrow[d, "\delta^K"] \\
 & \, \arrow[d, phantom, "\Downarrow H\alpha_\eta"] & HSA \arrow[r, "\alpha_{SA}"] \arrow[d, "Ha"'] \arrow[dr, phantom, "\Downarrow \alpha_a"] & KSA \arrow[d, "Ka"] \\
 & \, & HA \arrow[r, "\alpha_A"'] & KA
\end{tikzcd}
$$

$=$

$$
\begin{tikzcd}
HA \arrow[r, "\alpha_A"] \arrow[ddrr, "id"'] & KA \arrow[rr, "\eta_{KA}"] \arrow[drr, "K\eta_A"'] \arrow[ddrr, "id"'] & \, \arrow[r, phantom, "\Downarrow \overline{\eta}^K_A"] & SKA \arrow[d, "\delta^K_A"] \\
 & & \, \arrow[r, phantom, "\Downarrow K\alpha_\eta"] & KSA \arrow[d, "Ka"] \\
 & & HA \arrow[r, "\alpha_A"'] & KA
\end{tikzcd}
$$

For the pseudo-naturality part (2), the axiom for algebra 2-cells holds as

$$
\begin{tikzcd}
SHA \arrow[r, "S\alpha_A"] \arrow[d, "\delta^H_A"'] \arrow[dr, phantom, "\Downarrow \alpha^{*\,-1}_A"] & SKA \arrow[r, "SKf"] \arrow[d, "\delta^K_A"] \arrow[dr, phantom, "\Downarrow (\delta^K_f)^{-1}"] & SKB \arrow[d, "\delta^K_B"] \\
HSA \arrow[r, "\alpha_{SA}"'] \arrow[d, "Ha"'] \arrow[dr, phantom, "\Downarrow \alpha_a"] & KSA \arrow[r, "KSf"'] \arrow[d, "Ka"] \arrow[dr, phantom, "\Downarrow K\overline{f}_{a,b}"] & KSB \arrow[d, "Kb"] \\
HA \arrow[r, "\alpha_A"'] \arrow[dr, "Hf"'] & KA \arrow[r, "Kf"'] \arrow[d, phantom, "\Downarrow \alpha_f"] & KB \\
 & HB \arrow[ur, "\alpha_B"'] &
\end{tikzcd}
$$

$=$

$$
\begin{tikzcd}
 & SKA \arrow[dr, "SKf"] \arrow[d, phantom, "\Downarrow S\alpha_f"] & \\
SHA \arrow[ur, "S\alpha_A"] \arrow[r, "SHf"] \arrow[d, "\delta^H_A"'] \arrow[dr, phantom, "\Downarrow (\delta^H_f)^{-1}"] & SHB \arrow[r, "S\alpha_B"] \arrow[d, "\delta^H_B"] \arrow[dr, phantom, "\Downarrow \alpha^{*\,-1}_B"] & SKB \arrow[d, "\delta^K_B"] \\
HSA \arrow[r, "HSf"'] \arrow[d, "Ha"'] \arrow[dr, phantom, "\Downarrow H\overline{f}_{a,b}"] & HSB \arrow[r, "\alpha_{SB}"'] \arrow[d, "Hb"] \arrow[dr, phantom, "\Downarrow \alpha_b"] & KSB \arrow[d, "Kb"] \\
HA \arrow[r, "Hf"'] & HB \arrow[r, "\alpha_B"'] & KB
\end{tikzcd}
$$

by the pseudo-naturality of α and then by the axiom for modification α^*. This proves the proposition. $\square$

5.4.3 2-cells

Proposition 5.21. *For any modification $\gamma : \alpha \to \beta : H \to K$, where H and K are pseudo-endofunctors on $\mathbb{C}$, the following hold: if γ is a 2-cell in Ps-Dist^S and $\gamma : \alpha^* \to \beta^* : \delta^H \to \delta^K$, then γ is a 2-cell in Lift$_{\text{Ps-S-Alg}}$ and $\gamma : \Psi(\alpha^*) \to \Psi(\beta^*)$.*

Proof. The assumption requires that γ satisfy the axiom (5.2). Then the condition (5.6) for γ to be a 2-cell $\gamma : \Psi(\alpha^*) \to \Psi(\beta^*)$ in $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$ follows from that axiom (5.2) and that for modifications. □

Now we define a 2-functor Ψ: define Ψ_0 using Ψ^H defined in Proposition 5.19, as $\Psi_0(H,\delta^H) = (H,\Psi^H(\delta^H))$. For 1-cells, given a 1-cell (α,α^*) in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ we define $\Psi(\alpha,\alpha^*) = (\alpha,\Psi(\alpha^*))$ as in Proposition 5.20. And, for any 2-cell ζ in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$, we define $\Psi(\zeta) = \zeta$, again from Proposition 5.21

Proposition 5.22. *Ψ defined as above is a 2-functor.*

Proof. Follows from routine calculation. □

5.5 Proving the equivalence

We have constructed 2-functors Φ and Ψ,

$$\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{Ps\text{-}Dist}^{\mathsf{S}}$$

and now we are going to show that they define an equivalence of 2-categories.

Theorem 5.23. *The 2-categories $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$ are equivalent.*

Proof. First, we show that there exists a 2-natural isomorphism

$$\theta : Id \longrightarrow \Psi \circ \Phi.$$

Its component at $\widehat{H}$ is an arrow in $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$, i.e., a pseudo-natural transformation, but in this case in fact a 2-natural transformation

$$\theta_{\widehat{H}} : \widehat{H} \longrightarrow \Psi(\Phi(\widehat{H})),$$

whose component at a pseudo-algebra $\langle A, a, a_\mu, a_\eta \rangle$ is a pseudo-algebra map

$$(\theta_{\widehat{H}})_{\langle A,a,a_\mu,a_\eta\rangle} : \langle HA, \widehat{a}^{\widehat{H}}, \widehat{a}_\mu^{\widehat{H}}, \widehat{a}_\eta^{\widehat{H}} \rangle \to \langle HA, Ha \circ \widehat{\mu}_A^{\widehat{H}} \circ SH\eta_A, \widehat{a}_\mu^{\Psi(\Phi(\widehat{H}))}, \widehat{a}_\eta^{\Psi(\Phi(\widehat{H}))} \rangle$$

given as $(\theta_{\widehat{H}})_{\langle A,a,a_\mu,a_\eta\rangle} = (id_{HA}, \overline{(\theta_{\widehat{H}})_{A,a}})$, where the invertible 2-cell is given as

$$\begin{tikzcd}
 & SHSA \arrow[r, "\widehat{\mu}^{\widehat{H}}_A"] \arrow[d, "SHa"'] \arrow[dr, phantom, "\Downarrow \widehat{H}(\overline{a}_{a,\mu_A})^{-1}"] & HSA \arrow[d, "Ha"] \\
SHA \arrow[ur, "SH\eta_A"] \arrow[r, "id_{SHA}"'] \arrow[ur, phantom, "\Downarrow SHa_\eta", near end] & SHA \arrow[r, "\widehat{a}^{\widehat{H}}"'] & HA
\end{tikzcd}$$

We need to verify that this satisfies the axioms for pseudo-algebra maps; for the unit axiom, it follows from the unit axiom for pseudo-map $\widehat{H}(\overline{a}_{a,\mu_A})$ and the axiom for modification a_η. For the associativity axiom, it follows from, in the order of application, the modification axiom for a_η, the associativity axiom for pseudo-algebra map $\widehat{H}(\overline{a}_{a,\mu_A})$, the fact that Ha_η is an algebra 2-cell, and the right unit law for pseudo-algebra a. The 2-naturality of $\theta_{\widehat{H}}$ is proved by the fact that for any pseudo-map $(f, \overline{f}_{a,b})$, the 2-cell $H\overline{f}_{a,b}$ is an algebra 2-cell, and the unit law for pseudo-map $(f, \overline{f}_{a,b})$. The 2-naturality of θ follows from that of α and the fact that α lifts (α_a is an algebra 2-cell).

For the opposite direction, we construct a 2-natural isomorphism

$$\varepsilon : \Phi \circ \Psi \longrightarrow Id,$$

whose component of ε at δ^H is an invertible modification

$$\varepsilon_{\delta^H} : \Phi(\Psi(\delta^H)) \longrightarrow \delta^H$$

which is an arrow in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. This is a special case of a pseudo-distributive law over a 2-natural transformation, that is, over id_H. Its component $(\varepsilon_{\delta^H})_A$ at A is of type

$$(\varepsilon_{\delta^H})_A : H\mu_A \circ \delta^H_{SA} \circ SH\eta_A \longrightarrow \delta^H_A$$

and defined as in

$$\begin{tikzcd}
SHSA \arrow[r, "\delta^H_{SA}"] \arrow[dr, phantom, "\Downarrow \delta^{-1}_{\eta_A}"] & HS^2A \arrow[dr, "H\mu_A"] & \\
SHA \arrow[u, "SH\eta_A"] \arrow[r, "\delta^H_A"'] & HSA \arrow[u, "HS\eta_A"'] \arrow[r, "id_{HSA}"'] \arrow[ur, phantom, "\Downarrow H\lambda_A", near end] & HSA
\end{tikzcd}$$

This is indeed a pseudo-distributive law over id_H: we need to verify the axioms (α^*-1), (α^*-2) and the one for modifications. These follow from routine calculation. The 2-naturality of ε also follows from easy calculation. The θ and ε defined in this way are obviously isomorphic (i.e., invertible). This implies that Φ and Ψ are equivalences of 2-categories between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$. $\square$

Chapter 6

Composing Pseudo-Distributive Laws

In this chapter we further extend the discussion about composition of lifting and composition of distributive laws in Sections 3.6 and 4.6 and obtain similar results for the pseudo case. We first show that there are strict monoidal structures on the categories $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{S\text{-}Alg}$ and that the functors Θ and Ξ preserve those structures. We then consider corresponding structures in the pseudo setting. In $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, the situation is more complex owing to the pseudo-ness. We study the 2-category $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ to explain the special monoidal structure (bimonoidal structure), of which the structures on $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ are instances. This chapter bridges the constructions for pseudo-endofunctors and pseudo-monads, hence connecting the discussion in Chapter 5 to that of Chapter 7.

In Section 6.1 we start where we ended in Section 3.6. The construction for $\widehat{H}^2$ and $H\delta \circ \delta H : SH^2 \to H^2S$ both generalise to tensor products on $\mathsf{Lift}_{S\text{-}Alg}$ and $\mathsf{Dist}^{\mathsf{S}}$, respectively, providing strict monoidal structures on those categories, and the isomorphisms Θ and Ξ preserve those structures. We do not examine the case of $\mathsf{Dist}_{\mathsf{T}}$ and $\mathsf{Ext}_{Kl(T)}$ in this thesis but similar results hold there by duality too.

The structure on the 2-category $\mathsf{Ps\text{-}Endo}(\mathbb{C})$, consisting of pseudo-endofunctors on $\mathbb{C}$, pseudo-natural transformations between them, and modifications, is a *bimonoidal* structure. We consider the 2-categories $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ and bimonoidal structure on those 2-categories. We also prove the 2-functors Φ and Ψ preserve the bimonoidal structures. We then conclude the chapter by stating results generalising those in Section 3.6; we will use them in the next chapter when we consider pseudo-distributivity over pseudo-monads.

6.1 Monoidal structure on Dist^S

The following proposition describes the strict monoidal structure on Dist^S.

Proposition 6.1. *The following data yields a strict monoidal structure on the category Dist^S: Define a tensor $\otimes$ on Dist^S*

$$\otimes : \mathsf{Dist}^S \times \mathsf{Dist}^S \to \mathsf{Dist}^S$$

by

$$(H, \delta^H : SH \to HS) \otimes (K, \delta^K : SK \to KS) = (KH, \delta^{KH} : SKH \to KHS),$$

where $\delta^{KH} = K\delta^H \circ \delta^K H$, and, for arrows $\alpha : (H, \delta^H) \to (H', \delta^{H'} : SH' \to H'S)$ and $\beta : (K, \delta^K) \to (K', \delta^{K'} : SK' \to K'S)$. The value $\beta \otimes \alpha : (KH, \delta^{KH}) \to (K'H', \delta^{K'H'})$ is given by the horizontal composition $\beta\alpha : KH \to K'H'$. The unit for this tensor is $(Id_{\mathbb{C}}, id_S : S \to S)$.

Proof. Well-definedness of the object part follows by a slight generalisation of Lemma 3.20. For the arrow part, to see that it is well-defined, i.e., that S distributes over this natural transformation, we examine the diagram

$$\begin{tikzcd}
SKH \arrow[r, "\delta^K H"] \arrow[d, "S\beta\alpha"'] & KSH \arrow[r, "K\delta^H"] \arrow[d, "\beta S\alpha"] & KHS \arrow[d, "\beta\alpha S"] \\
SK'H' \arrow[r, "\delta^{K'}H'"'] & K'SH' \arrow[r, "K'\delta^{H'}"'] & K'H'S
\end{tikzcd}$$

which commutes since S distributes over both α and β. It is easy to see that this tensor is strictly associative and has a strict unit $(Id_{\mathbb{C}}, id_S : S \to S)$, making Dist^S a strict monoidal category. □

Similarly, strict monoidal structure on $\mathsf{Lift}_{S\text{-}Alg}$ is given as follows:

Proposition 6.2. *The following data yields a strict monoidal structure on the category* $\mathsf{Lift}_{S\text{-}Alg}$*: we define a tensor* $\otimes$ *on the category* $\mathsf{Lift}_{S\text{-}Alg}$

$$\otimes : \mathsf{Lift}_{S\text{-}Alg} \times \mathsf{Lift}_{S\text{-}Alg} \to \mathsf{Lift}_{S\text{-}Alg}$$

as

$$(H,\widehat{H}) \otimes (K,\widehat{K}) = (KH, \widehat{K}\widehat{H})$$

on objects, and for arrows $\alpha : (H,\widehat{H}) \to (H',\widehat{H}')$ *and* $\beta : (K,\widehat{K}) \to (K',\widehat{K}')$*, the composite* $\beta \otimes \alpha : (KH, \widehat{K}\widehat{H}) \to (K'H', \widehat{K}'\widehat{H}')$ *is again given by horizontal composition* $\beta\alpha : KH \to K'H'$*. The unit for the tensor is* $(Id_{\mathbb{C}}, Id_{S\text{-}Alg})$*.*

Proof. The well-definedness of the object part is immediate. To see that $\beta\alpha$ lifts to S-Alg from $\widehat{K}\widehat{H}$ to $\widehat{K}'\widehat{H}'$, we need to verify that, for any S-algebra $\langle A, a\rangle$, the component $(\beta\alpha)_A$ is an S-algebra map from $\langle KHA, \widehat{a}^{\widehat{K}\widehat{H}}\rangle$ to $\langle K'H'A, \widehat{a}^{\widehat{K}'\widehat{H}'}\rangle$. By the equation (3.7), the structure arrow decomposes as

$$\widehat{a}^{\widehat{K}\widehat{H}} = \widehat{\widehat{a}^{\widehat{H}}}^{\widehat{K}} = K\widehat{a}^{\widehat{H}} \circ \widehat{\mu}^{\widehat{K}}_{HA} \circ SK\eta_{HA}, \tag{6.1}$$

similarly for the arrow $\widehat{a}^{\widehat{K}'\widehat{H}'}$. Therefore well-definedness amounts to the commutativity of the following diagram:

$$\begin{tikzcd}
SKHA \arrow[r, "SK\alpha_A"] \arrow[d, "SK\eta_{HA}"'] & SKH'A \arrow[r, "S\beta_{H'A}"] \arrow[d, "SK\eta_{H'A}"] & SK'H'A \arrow[d, "SK'\eta_{H'A}"] \\
SKSHA \arrow[r, "SKS\alpha_A"] \arrow[d, "\widehat{\mu}^{\widehat{K}}_{HA}"'] & SKSH'A \arrow[r, "S\beta_{SH'A}"] \arrow[d, "\widehat{\mu}^{\widehat{K}}_{H'A}"] \arrow[dr, phantom, "(B)"] & SK'SH'A \arrow[d, "\widehat{\mu}^{\widehat{K}'}_{H'A}"] \\
KSHA \arrow[r, "KS\alpha_A"] \arrow[d, "K\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "(A)"] & KSH'A \arrow[r, "\beta_{SH'A}"] \arrow[d, "K\widehat{a}^{\widehat{H}'}"] & K'SH'A \arrow[d, "K'\widehat{a}^{\widehat{H}'}"] \\
KHA \arrow[r, "K\alpha_A"'] & KH'A \arrow[r, "\beta_{H'A}"'] & K'H'A
\end{tikzcd}$$

which commutes by the fact that α and β lift to S-Alg for the squares (A) and (B), respectively, and by the naturality of η, β and $\widehat{\mu}$ for the rest of the squares. Note that the component $(\beta\alpha)_A$ has two equivalent decompositions $\beta_{H'A} \circ K\alpha_A$ and $K'\alpha_A \circ \beta_{HA}$, and in the above diagram we chose the former. The composition is obviously strictly associative and its unit is $(Id_{\mathbb{C}}, Id_{S\text{-}Alg})$, therefore $\mathsf{Lift}_{S\text{-}Alg}$ is again a strict monoidal category. $\square$

We have seen in Section 5.5 that the categories $\mathsf{Lift}_{S\text{-}Alg}$ and $\mathsf{Dist}^{\mathsf{S}}$ are equivalent. The following theorem says that they are also equivalent as strict monoidal categories, hence the equivalences Θ and Ξ preserve the structures described above.

Theorem 6.3. *The functors constructed in Chapter 5*

$$\mathsf{Lift}_{S\text{-}Alg} \underset{\Xi}{\overset{\Theta}{\rightleftarrows}} \mathsf{Dist}^{S}$$

are strict monoidal functors.

Proof. Given two objects $(H,\widehat{H})$ and $(K,\widehat{K})$ of $\mathsf{Lift}_{S\text{-}Alg}$, the value of the functor Θ at $(H,\widehat{H})\otimes(K,\widehat{K}) = (KH,\widehat{K}\widehat{H})$ is a distributive law $\Theta(\widehat{K}\widehat{H})$, which, by definition of Θ, the equation (6.1) and the naturality of $\widehat{\mu}^{\widehat{K}}$, yields

$$\begin{aligned}
\Theta(\widehat{K}\widehat{H}) &= \widehat{\mu}^{\widehat{K}\widehat{H}} \circ SKH\eta \\
&= K\widehat{\mu}^{\widehat{H}} \circ \widehat{\mu}^{\widehat{K}}HS \circ SK\eta HS \circ SKH\eta \\
&= K\widehat{\mu}^{\widehat{H}} \circ KSH\eta \circ \widehat{\mu}^{\widehat{K}}H \circ SK\eta H \\
&= K\Theta(\widehat{H}) \circ \Theta(\widehat{K})H \\
&= \Theta(\widehat{H}) \otimes \Theta(\widehat{K})
\end{aligned}$$

demonstrating that Θ preserves the tensor $\otimes$. Since Θ and Ξ are isomorphisms (Theorem 3.19), it means that they define an isomorphism of strict monoidal categories. $\square$

Although it follows from the above, a direct proof that Ξ is strict monoidal goes as follows: let $(H,\delta^H : SH \to HS)\otimes(K,\delta^K : SK \to KS) = (KH,\delta^{KH} : SKH \to KHS)$ for objects $(H,\delta^H : SH \to HS)$ and $(K,\delta^K : SK \to KS)$ of $\mathsf{Dist}^{\mathsf{S}}$. Then, for the lifting $\Xi(\delta^H)\otimes\Xi(\delta^K) = \Xi(\delta^K)\Xi(\delta^H)$ the structure map $\widehat{a}^{\Xi(\delta^K)\Xi(\delta^H)}$ of the value at an S-algebra $\langle A,a\rangle$ is calculated as follows:

$$\begin{aligned}
\widehat{a}^{\Xi(\delta^K)\Xi(\delta^H)} &= K\widehat{a}^{\Xi(\delta^H)} \circ \widehat{\mu}_{HA}^{\Xi(\delta^K)} \circ SK\eta_{HA} \\
&= KHa \circ K\delta_A^H \circ K\mu_{HA} \circ \delta_{SHA}^K \circ SK\eta_{HA} \\
&= KHa \circ K\delta_A^H \circ K\mu_{HA} \circ KS\eta_{HA} \circ \delta_{HA}^K \\
&= KHa \circ K\delta_A^H \circ \delta_{HA}^K \\
&= KHa \circ (K\delta^H \circ \delta^K H)_A \\
&= \widehat{a}^{\Xi(\delta^{KH})}
\end{aligned}$$

showing that Ξ preserves the tensor on $\mathsf{Dist}^{\mathsf{S}}$. It follows that, since for the units $\Theta(Id_{\mathbb{C}}, id_{S\text{-}Alg}) = \mu \circ S\eta = id_S$ and $\widehat{a}^{\Xi(id_S)} = a$ hold, Θ and Ξ are strict monoidal functors between $\mathsf{Lift}_{S\text{-}Alg}$ and $\mathsf{Dist}^{\mathsf{S}}$.

For any category $\mathcal{C}$, the category of endofunctors on $\mathcal{C}$ and natural transformations between them is a strict monoidal category with composition of endofunctors as the tensor and the identity functor on $\mathcal{C}$ as the unit. Now, let U_1 and U_2 be the forgetful functors from $\mathsf{Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{S\text{-}Alg}$ to $\mathsf{Endo}(\mathcal{C})$, respectively, which send each object to its first component. Then the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Dist}^{\mathsf{S}} \arrow[rr, "\Xi"] \arrow[dr] & & \mathsf{Lift}_{S\text{-}Alg} \arrow[dl] \\
& \mathsf{Endo}(\mathcal{C}) &
\end{tikzcd} \tag{6.2}$$

Corollary 6.4. *The diagram* (6.2) *is a diagram of strict monoidal categories and strict monoidal functors.*

6.2 The structure on Ps-Dist$^{\mathsf{S}}$

In this section we consider the structure discussed in the previous section in the pseudo-setting, where the strict monoidal structure is replaced by something more complex. First we take a 2-category $\mathbb{C}$ and consider the 2-category $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ of pseudo-endofunctors on $\mathbb{C}$, pseudo-natural transformations between them, and modifications between them. This category does not have a strict monoidal structure like $\mathsf{Endo}(\mathcal{C})$ in the previous section; moreover, neither does it have monoidal structure, mainly because of the pseudo-ness. The structure can be called a bimonoidal structure, the precise meaning of which we detail later in the section. The 2-categories $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ have the same structures on them, which are preserved by the equivalence pseudo-functors Φ and Ψ in Chapter 5, making a diagram similar to (6.2) commute.The definition of bimonoidal bicategory can be found in Section 2.5.

6.2.1 The 2-category of pseudo-endofunctors *Ps-Endo*$(\mathbb{C})$

We first have a look at the structure on the 2-category $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ to describe how the strict monoidal structure in the ordinary (non-pseudo) case is replaced by the bi-monoidal structure in the pseudo-case. The category $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ is a 2-category with the composition of 1-cells in $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ given by the vertical composition of pseudo-natural transformations, which is well-defined up to an equality because the composition of 1-cells in the 2-category $\mathbb{C}$ is defined up to an equality and is associative, and also because the pasting of 2-cells in $\mathbb{C}$ is associative. For 2-cells in $\mathsf{Ps\text{-}Endo}(\mathbb{C})$, i.e., modifications in $\mathbb{C}$, both the horizontal and vertical compositions are well-defined

There exists a tensor on $\mathsf{Ps\text{-}Endo}(\mathbb{C})$,

$$\otimes : \mathsf{Ps\text{-}Endo}(\mathbb{C}) \times \mathsf{Ps\text{-}Endo}(\mathbb{C}) \to \mathsf{Ps\text{-}Endo}(\mathbb{C}),$$

which is a pseudo-functor rather than a functor. The 0-cell part of this tensor is given by the composition of pseudo-endofunctors, just as the case in $\mathsf{Endo}(\mathcal{C})$, and its associativity is guaranteed by the coherence axiom of pseudo-functors. However, this is not the case for the 1-cells. The arrow part of the tensor $\otimes$ on $\mathsf{Endo}(\mathcal{C})$ is given by the horizontal composition of natural transformations, which is obviously associative. But that is not true for $\mathsf{Ps\text{-}Endo}(\mathbb{C})$. For $\mathsf{Ps\text{-}Endo}(\mathbb{C})$, the 1-cells in $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ are pseudo-natural transformations, for which horizontal composition is defined only up to an invertible modification. Let $\alpha : H \to H'$ and $\beta : K \to K'$ to be pseudo-natural transformations between pseudo-endofunctors on $\mathbb{C}$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "H"{name=U}] \arrow[r, bend right, "H'"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow\alpha"] & \mathbb{C} \arrow[r, bend left, "K"{name=U2}] \arrow[r, bend right, "K'"'{name=D2}] \arrow[phantom, from=U2, to=D2, "\Downarrow\beta"] & \mathbb{C} \end{tikzcd}$$

Then, the arrow part of the tensor $\alpha \otimes \beta : KH \to K'H'$ can be defined as $\beta H' \circ K\alpha$ or $K'\alpha \circ \beta H$, which are equal only up to an invertible modification

$$\begin{tikzcd} KH \arrow[r, "K\alpha"] \arrow[d, "\beta H"'] \arrow[dr, phantom, "\Downarrow\beta_\alpha"] & KH' \arrow[d, "\beta H'"] \\ K'H \arrow[r, "K'\alpha"'] & K'H'. \end{tikzcd}$$

Let us now assume that we choose $\beta H' \circ K\alpha$ to be the 1-cell part of the tensor. (In fact, it does not affect the later discussion what choice we make.) Suppose we also have a pseudo-natural transformation $\gamma : L \to L'$, with $L, L' : \mathbb{C} \to \mathbb{C}$. Then the diagram

$$\begin{tikzcd}
\alpha \otimes \beta \otimes \gamma \arrow[r, "\otimes \times Id"] \arrow[d, "Id \times \otimes"'] & (\beta H' \circ K\alpha) \otimes \gamma \arrow[d, "\otimes"] \\
\alpha \otimes (\gamma K' \circ L\beta) \arrow[r, "\otimes"'] & ?
\end{tikzcd}$$

need not be commutative because $(\gamma K' \circ L\beta)H' \circ LK\alpha$ and $\gamma K' H' \circ L(\beta H' \circ K\alpha)$ are equal only up to an invertible 2-cell because the functoriality of pseudo-functors hold only up to invertible 2-cells. The situation is similar for units, as we have an equality for one of the unit holds, but not for the other: for 1-cells $\alpha : H \to H', id_H : H \to H$ and $id_{H'} : H' \to H'$, one only has $id_H \otimes \alpha \cong \alpha H \circ H id_H$, but has $\alpha \otimes id_{H'} = id_{H'} H' \circ H\alpha$.

Proposition 6.5. *The following data defines a pseudo-functor*

$$\otimes : \textit{Ps-Endo}(\mathbb{C}) \times \textit{Ps-Endo}(\mathbb{C}) \to \textit{Ps-Endo}(\mathbb{C})$$

- *given a pair H and K of 0-cells, the value $H \otimes K$ is defined to be KH,*
- *given a pair $\alpha : H \to K$ and $\beta : H' \to K'$ of 1-cells, choose $\beta H' \circ K\alpha$ as the value $\alpha \otimes \beta$.*
- *given $\zeta : \alpha \to \alpha' : H \to K$ and $\xi : \beta \to \beta' : H' \to K'$, the value $\zeta \otimes \xi : \alpha \otimes \beta \to \alpha' \otimes \beta'$ is given by the horizontal composition $\xi H' \otimes K\zeta$.*

This is not a Gray-monoid because here we are dealing with pseudo-functors rather than 2-functors: in order to be a Gray-monoid, one would need $- \otimes H$ to be a 2-functor, but in general, it is not. We do not need to develop a precise general statement of the structure for the purpose of the thesis, i.e., the development of pseudo-distributive laws, but for completeness, we state the following. The definition of bimonoidal bicategory appears as Definition 2.28. Here, a bimonoidal 2-category is a bimonoidal bicategory whose tensor product is a pseudo-functor and, whose underlying bicategory is a 2-category.

Proposition 6.6. *Ps-Endo$(\mathbb{C})$ is a bimonoidal 2-category.*

Proof. The proof an immediate consequence of [GPS95]. ☐

6.2.2 The structure on $\mathsf{Lift}_{\textit{Ps-S-Alg}}$

Now we turn to the 2-category $\mathsf{Lift}_{\textit{Ps-S-Alg}}$. This 2-category has a structure that is essentially the same as $\mathsf{Ps\text{-}Endo}(\mathbb{C})$. The structure is the pseudo-version of that on $\mathsf{Lift}_{\textit{S-Alg}}$ we saw in Section 6.1, but similarly to the case of $\mathsf{Ps\text{-}Endo}(\mathbb{C})$, owing to the pseudoness we lose not only the strictness but also the equality on the composition of 1-cells: the composition of 1-cells in $\mathsf{Lift}_{\textit{Ps-S-Alg}}$ is defined only up to invertible 2-cells as explained later. Moreover, the similarity between $\mathsf{Lift}_{\textit{Ps-S-Alg}}$ and $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ extends to the fact that, for 0-cells, the structure on $\mathsf{Lift}_{\textit{Ps-S-Alg}}$ is strictly associative because the 0-cell part of the tensor, which we define below, is defined in terms of the composition of pseudo-functors. This is not the case, however, for the structure on the 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$, which we investigate in the next section.

Define a tensor $\otimes$ on $\mathsf{Lift}_{\textit{Ps-S-Alg}}$ as a pseudo-functor

$$\otimes : \mathsf{Lift}_{\textit{Ps-S-Alg}} \times \mathsf{Lift}_{\textit{Ps-S-Alg}} \to \mathsf{Lift}_{\textit{Ps-S-Alg}},$$

such that, for a pair of 0-cells $(H,\widehat{H})$ and $(K,\widehat{K})$, the value $(H,\widehat{H})\otimes(K,\widehat{K})$ is given by the composition of pseudo-endofunctors, i.e.

$$(H,\widehat{H})\otimes(K,\widehat{K}) = (KH,\widehat{K}\widehat{H}).$$

And, given 1-cells $(\alpha,\widehat{\alpha}) : (H,\widehat{H}) \to (H',\widehat{H}')$ and $(\beta,\widehat{\beta}) : (K,\widehat{K}) \to (K',\widehat{K}')$, they are sent to

$$(\alpha,\widehat{\alpha})\otimes(\beta,\widehat{\beta}) = (\alpha\otimes\beta,\widehat{\alpha\otimes\beta}),$$

where $\widehat{\alpha\otimes\beta}$ is a pseudo-natural transformation from $\widehat{K}\widehat{H}$ to $\widehat{K}'\widehat{H}'$, which means, firstly, for each pseudo-S-algebra $\langle A,a,a_\mu,a_\eta\rangle$, which we abbreviate as $\langle A,a\rangle$, the component $(\alpha\otimes\beta)_A = (\beta H'\circ K\alpha)_A$ constitute a pseudo-algebra map from $\langle KHA,\widehat{a}^{\widehat{K}\widehat{H}}\rangle$ to

$\langle K'H'A, \widehat{a}^{\widehat{K}'\widehat{H}'}\rangle$, together with the 2-cell $\overline{(\beta H' \circ K\alpha)_{A,a}}$

$$\begin{tikzcd}
SKHA \arrow[r, "SK\alpha_A"'] \arrow[d, "SK\eta_{HA}"'] \arrow[dr, phantom, "\Downarrow SK\eta^{-1}_{\alpha_A}"] & SKH'A \arrow[r, "S\beta_{H'A}"'] \arrow[d, "SK\eta_{H'A}"] \arrow[dr, phantom, "\Downarrow S\beta_{\eta_{H'A}}"] & SK'H'A \arrow[d, "SK'\eta_{H'}A"] \\
SKSHA \arrow[r, "SKS\alpha_A"] \arrow[d, "\widehat{\mu}^{\widehat{K}}_{HA}"'] \arrow[dr, phantom, "\Downarrow {\widehat{\mu}^{\widehat{K}}}^{-1}_{\alpha_A}"] & SKSH'A \arrow[r, "S\beta_{SH'A}"] \arrow[d, "\widehat{\mu}^{\widehat{K}}_{H'A}"] \arrow[dr, phantom, "\Downarrow \overline{\beta_{SH'A,\mu_{H'A}}}"] & SK'SH'A \arrow[d, "\widehat{\mu}^{\widehat{K}'}_{H'}A"] \\
KSHA \arrow[r, "KS\alpha_A"] \arrow[d, "K\widehat{a}^{\widehat{H}}"'] \arrow[dr, phantom, "\Downarrow K\overline{\alpha_{A,a}}"] & KSH'A \arrow[r, "\beta_{SH'A}"] \arrow[d, "K\widehat{a}^{\widehat{H}'}"] \arrow[dr, phantom, "\Downarrow \beta_{\widehat{a}^{\widehat{H}'}}"] & K'SH'A \arrow[d, "K'\widehat{a}^{\widehat{H}'}"] \\
KHA \arrow[r, "K\alpha_A"] & KH'A \arrow[r, "\beta_{H'A}"] & K'H'A
\end{tikzcd} \tag{6.3}$$

that satisfies the axioms for pseudo-algebra maps. Secondly, $\beta H' \circ K\alpha$ should be pseudo-natural as pseudo-algebra maps. Given any pseudo-algebra map $(f, \overline{f}_{a,b}) : \langle A, a, a_\mu, a_\eta\rangle \to \langle B, b, b_\mu, b_\eta\rangle$, the second component of the value $\widehat{K}\widehat{H}(f, \overline{f}_{a,b})$ is given by the diagram

$$\begin{tikzcd}
SKHA \arrow[r, "SKHf"'] \arrow[d, "SK\eta_{HA}"'] \arrow[dr, phantom, "\Downarrow SK\eta^{-1}_{Hf}"] & SKHB \arrow[d, "SK\eta_{HB}"] \\
SKSHA \arrow[r, "SKSHf"] \arrow[d, "\widehat{\mu}^{\widehat{K}}_{HA}"'] \arrow[dr, phantom, "\Downarrow {\widehat{\mu}^{\widehat{K}}_{Hf}}^{-1}"] & SKSHB \arrow[d, "\widehat{\mu}^{\widehat{K}}_{HB}"] \\
KSHA \arrow[r, "KSHf"] \arrow[d, "K\widehat{a}^{\widehat{K}}"'] \arrow[dr, phantom, "\Downarrow K\widehat{H}(\overline{f}_{a,b})"] & KSHB \arrow[d, "K\widehat{b}^{\widehat{K}}"] \\
KHA \arrow[r, "KHf"] & KHB
\end{tikzcd}$$

The proof that the axiom (5.5) holds for this 2-cell is given, in the order of application, by pseudo-naturality of β, that of $\widehat{\beta}$, then by pseudo-naturality of $\widehat{\mu}^{K}$ and β, and then by that of η.

And for the 2-cells, given 2-cells $\zeta : (\alpha, \widehat{\alpha}) \to (\alpha', \widehat{\alpha}') : (H, \widehat{H}) \to (H', \widehat{H}')$ and $\xi : (\beta, \widehat{\beta}) \to (\beta', \widehat{\beta}') : (K, \widehat{K}) \to (K', \widehat{K}')$, the value $\zeta \otimes \xi$ is given by the composite modification $\xi H' \circ K\zeta$, which obviously lifts from $\widehat{\alpha} \otimes \widehat{\beta}$ to $\widehat{\alpha}' \otimes \widehat{\beta}'$.

Putting all these together,

Proposition 6.7. *The data described above define a pseudo-functor*

$$\otimes : \mathit{Lift}_{Ps\text{-}S\text{-}Alg} \times \mathit{Lift}_{Ps\text{-}S\text{-}Alg} \longrightarrow \mathit{Lift}_{Ps\text{-}S\text{-}Alg}.$$

Proof. Functoriality is verified routinely. □

Our next claim (although we do not really need this for our analysis) is that this tensor $\otimes$ gives a bimonoidal structure on $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$. On 0-cells the tensor $\otimes$ is strictly associative, which follows from the fact that the composition of pseudo-endofunctors is strictly associative. However, for the 1-cells, the situation is just as the case for $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ and again is not very simple. We have associativity only up to coherent invertible 2-cells, thereby failing to make $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$ a monoidal category.

Proposition 6.8. $\mathit{Lift}_{Ps\text{-}S\text{-}Alg}$ *is a bimonoidal 2-category.*

Proof. Follows routinely from above and Proposition 6.6. □

6.2.3 The structure on $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$

Similarly to the cases of $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, we define the tensor

$$\otimes : \mathsf{Ps\text{-}Dist}^{\mathsf{S}} \times \mathsf{Ps\text{-}Dist}^{\mathsf{S}} \to \mathsf{Ps\text{-}Dist}^{\mathsf{S}},$$

as a pseudo-functor such that, given a pair of 0-cells, $(H, (\delta^H, \overline{\mu}^H, \overline{\eta}^K))$ and $(K, (\delta^K, \overline{\mu}^K, \overline{\eta}^K))$, the value $(H, (\delta^H, \overline{\mu}^H, \overline{\eta}^K)) \otimes (K, (\delta^K, \overline{\mu}^K, \overline{\eta}^K))$ is defined to be $(KH, (\delta^{KH}, \overline{\mu}^{KH}, \overline{\eta}^{KH}))$, where δ^{KH} is a pseudo-natural transformation

$$K\delta^H \circ \delta^K H : SKH \to KHS,$$

and the invertible modifications $\overline{\mu}^{KH}$ is given by:

$$\begin{tikzcd}
S^2KH \arrow[r, "S\delta^K H"] \arrow[dd, "\mu KH"'] & SKSH \arrow[r, "SK\delta^H"] \arrow[dr, "\delta^K SH"'] \arrow[d, phantom, "\Downarrow \overline{\mu}^K H"] & SKHS \arrow[r, "\delta^K HS"] \arrow[d, phantom, "\Downarrow \delta^{K\,-1}_{\delta^H}"] & KSHS \arrow[r, "K\delta^H S"] \arrow[d, phantom, "\Downarrow K\overline{\mu}^H"] & KHS^2 \arrow[dd, "KH\mu"] \\
 & & KS^2H \arrow[ur, "KS\delta^H"'] \arrow[d, "K\mu H"] & & \\
SKH \arrow[rr, "\delta^K H"] & & KSH \arrow[rr, "K\delta^H"] & & KHS
\end{tikzcd}$$

and $\overline{\eta}^{KH}$ by:

$$\begin{tikzcd}
KH \arrow[rr, "\eta KH"] \arrow[drr, "K\eta H"'] \arrow[ddrr, "KH\eta"'] & & SKH \arrow[d, "\delta^K H"] \arrow[dl, phantom, "\Downarrow \overline{\eta}^K H"] \\
 & & KSH \arrow[d, "K\delta^H"] \arrow[dl, phantom, "\Downarrow K\overline{\eta}^H"] \\
 & & KHS
\end{tikzcd}$$

Proposition 6.9. *The data* $(\delta^{KH}, \overline{\mu}^{KH}, \overline{\eta}^{KH})$ *defined as above yields a pseudo-distributive law over* KH.

Proof. We need to verify that the two invertible modifications $\overline{\mu}^{KH}$ and $\overline{\eta}^{KH}$ satisfy the three axioms for pseudo-distributive laws. The proof that axioms (H-1)-(H-3) hold for these two modifications are given by: in the order of application, (H-1) by axiom (1) for δ (twice) and pseudo-naturality of δ, (H-2) by axiom (2) for δ (twice) and the axiom for modification $\overline{\eta}$, and (H-3) by pseudo-naturality of δ, axiom (3) (twice) and the modification axiom for $\overline{\mu}$. $\square$

Now we are left to define the 1-cell and 2-cell parts of the tensor $\otimes$ on Ps-Dist^S. Given a pair of 1-cells,

$$(\alpha, \alpha^*) : (H, (\delta^H, \overline{\mu}^H, \overline{\eta}^H)) \to (H', (\delta^{H'}, \overline{\mu}^{H'}, \overline{\eta}^{H'})),$$

and

$$(\beta, \beta^*) : (K, (\delta^K, \overline{\mu}^K, \overline{\eta}^K)) \to (K', (\delta^{K'}, \overline{\mu}^{K'}, \overline{\eta}^{K'})),$$

we define $(\alpha, \alpha^*) \otimes (\beta, \beta^*) = (\alpha \otimes \beta, (\alpha \otimes \beta)^*)$, where $\alpha \otimes \beta$ is $\beta H' \circ K\alpha$, and the invertible modification $(\alpha \otimes \beta)^*$ is defined as

$$\begin{tikzcd}
SKH \arrow[r, "\delta^K H"] \arrow[d, "SK\alpha"'] \arrow[dr, phantom, "\Downarrow \delta^K_\alpha"] & KSH \arrow[r, "K\delta^H"] \arrow[d, "KS\alpha"] \arrow[dr, phantom, "\Downarrow K\alpha^*"] & KHS \arrow[d, "K\alpha S"] \\
SKH' \arrow[r, "\delta^K H'"] \arrow[d, "S\beta H'"'] \arrow[dr, phantom, "\Downarrow \beta^* H'"] & KSH' \arrow[r, "K\delta^{H'}"] \arrow[d, "\beta_{SH'}"] \arrow[dr, phantom, "\Downarrow \beta^{-1}_{\delta^{H'}}"] & KH'S \arrow[d, "\beta H' S"] \\
SK'H' \arrow[r, "\delta^{K'} H'"] & K'SH' \arrow[r, "K'\delta^{H'}"] & K'H'S
\end{tikzcd}$$

Lemma 6.10. *The invertible modification* $(\alpha \otimes \beta)^*$ *defined in the diagram above is a pseudo-distributive law of* S *over* $\alpha \otimes \beta$ *with respect to* δ^{KH} *and* $\delta^{K'H'}$.

Proof. It is easy to see that the above diagram defines a modification. We now need to verify that $(\alpha \otimes \beta)^*$ satisfies axioms (α^*-1) and (α^*-2). For (α^*-1), the proof is routinely given by using the axioms for pseudo-distributive laws α^* and β^*, and for modifications $\overline{\mu}^H$, $\overline{\mu}^K$, $\overline{\eta}^H$ and $\overline{\eta}^K$. $\square$

Finally, for 2-cells: given 2-cells $\zeta : (\alpha, \alpha^*) \to (\alpha', \alpha'^*)$ and $\xi : (\beta, \beta^*) \to (\beta', \beta'^*)$ the value $\zeta \otimes \xi$ is defined to be $\xi H' \circ K\zeta$, which is easily seen to satisfy the condition that S distribute over it with respect to $\alpha \otimes \beta$ and $\alpha' \otimes \beta'$.

Now we can state the following:

Proposition 6.11. *The data described above defines a pseudo-functor*

$$\otimes : \mathit{Ps\text{-}Dist}^S \times \mathit{Ps\text{-}Dist}^S \to \mathit{Ps\text{-}Dist}^S.$$

Now we consider the bimonoidal structure on $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$. Recall that both on $\mathsf{Ps\text{-}Endo}(\mathbb{C})$ and $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, the tensor is strictly associative on 0-cells. However, for the case of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$, since the 0-cells are defined in terms of pseudo-natural transformations, the tensor is associative only up to invertible 2-cells. The situation for 1-cells and 2-cells are similar to the others.

Proposition 6.12. $\mathit{Ps\text{-}Dist}^S$ *is a bimonoidal 2-category.*

6.3 Equivalence of bimonoidal categories

Proposition 6.13. *The pseudo-functors* Φ *and* Ψ *constructed in Chapter 5*

$$\mathit{Ps\text{-}Dist}^S \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathit{Lift}_{Ps\text{-}S\text{-}Alg}$$

preserve the bimonoidal structures (not strictly), or in other words, there exist coherent isomorphisms

$$\Psi(-) \otimes \Psi(-) \cong \Psi(- \otimes -) \tag{6.4}$$

$$\Phi(-) \otimes \Phi(-) \cong \Phi(- \otimes -). \tag{6.5}$$

Proof. For Ψ: assume we are given pseudo-distributive laws $(\delta^H, \overline{\mu}^H, \overline{\eta}^K)$ and $(\delta^K, \overline{\mu}^K, \overline{\eta}^K)$. We need to show that there exists a pseudo-natural isomorphism

$$\Psi(\delta^H) \otimes \Psi(\delta^K) \cong \Psi(\delta^H \otimes \delta^K)$$

between the liftings of KH, which amounts to showing, for each pseudo-S-algebra $\langle A, a, a_\mu, a_\eta \rangle$, the existence of a pseudo-algebra map which consists of the identity map and an invertible 2-cell, satisfying the pseudo-naturality condition Here we show the construction of the pseudo-algebra map only. This pseudo-algebra map, from $\Psi(\delta^K)\Psi(\delta^H)\langle A, a, a_\mu, a_\eta \rangle$ to $\Psi(\delta^{KH})\langle A, a, a_\mu, a_\eta \rangle$ is the component at $\langle A, a, a_\mu, a_\eta \rangle$ of the pseudo-natural isomorphism. If we denote the structure maps of the pseudo-algberas $\Psi(\delta^K)\Psi(\delta^H)\langle A, a, a_\mu, a_\eta \rangle$ and $\Psi(\delta^{KH})\langle A, a, a_\mu, a_\eta \rangle$ by $\widehat{a}^{\Psi(\delta^K)\Psi(\delta^H)}$ and $\widehat{a}^{\Psi(\delta^{KH})}$, respectively, we can calculate them, from the definition of the pseudo-functor Ψ, as;

$$\begin{aligned} \widehat{a}^{\Psi(\delta^K)\Psi(\delta^H)} &= \widehat{\widehat{a}^{\Psi(\delta^H)}}^{\Psi(\delta^K)} = K(Ha \circ \delta^H_A) \circ \delta^K_{HA} \\ \widehat{a}^{\Psi(\delta^{KH})} &= \widehat{a}^{\Psi(K\delta^H \circ \delta^K H)} = KHa \circ K\delta^H_A \circ \delta^K_{HA} \end{aligned}$$

which are equal up to an invertible 2-cell because of the pseudo-functoriality of K. The pseudo-functoriality is defined in such a way that the conditions for this 2-cell to be a pseudo-map are satisfied automatically.

Similarly, for the opposite direction Φ, we only show the construction for the 0-cells. Given 0-cells $(H, \widehat{H})$ and $(K, \widehat{K})$ of $\mathsf{Lift}_{Ps\text{-}S\text{-}Alg}$, we show that there exists an invertible 2-cell that serves as a pseudo-distributive law of S over id_{KH} with respect to $\Phi(\widehat{H}) \otimes \Phi(\widehat{K})$ and $\Phi(\widehat{H} \otimes \widehat{K})$, which provides the pseudo-natural isomorphism between $\Phi(\widehat{H}) \otimes \Phi(\widehat{K}) \cong \Phi(\widehat{H} \otimes \widehat{K})$. By definition, the value on the right hand side is given by

$$\Phi(\widehat{H} \otimes \widehat{K}) = \widehat{\mu}^{\widehat{K}\widehat{H}} \circ SKH\eta = \widehat{\widehat{\mu}^{\widehat{H}}}^{\widehat{K}} \circ SKH\eta$$

which is, from Theorem 5.23 and the pseudo-naturality, equal up to an invertible 2-cell to

$$\begin{aligned} \widehat{\widehat{\mu}^{\widehat{H}}}^{\widehat{K}} \circ SKH\eta &\cong K\widehat{\mu}^{\widehat{H}} \circ \widehat{\mu}^{\widehat{K}} HS \circ SK\eta HS \circ SKH\eta \\ &\cong K\widehat{\mu}^{\widehat{H}} \circ \widehat{\mu}^{\widehat{K}} HS \circ SKSH\eta \circ SK\eta H \\ &\cong K\widehat{\mu}^{\widehat{H}} \circ KSH\eta \circ \widehat{\mu}^{\widehat{K}} H \circ SK\eta H \end{aligned}$$

On the other hand, we have

$$\Phi(\widehat{H}) \otimes \Phi(\widehat{K}) = K\Phi(\widehat{H}) \circ \Phi(\widehat{K})H = K(\widehat{\mu}^{\widehat{H}} \circ SH\eta) \circ \widehat{\mu}^{\widehat{K}}H \circ SK\eta H$$

which is, again up to the pseudo-functoriality, equal to the above. Checking coherence is routine but lengthy. □

The above discussion leads to the following theorem:

$$\begin{tikzcd} \mathsf{Ps\text{-}Dist}^{\mathsf{S}} \arrow[rr, "\Psi"] \arrow[dr, "U_1"'] & & \mathsf{Lift}_{Ps\text{-}S\text{-}Alg} \arrow[dl, "U_2"] \\ & \mathsf{Ps\text{-}Endo}(\mathbb{C}) & \end{tikzcd} \tag{6.6}$$

Theorem 6.14. *Diagram* (6.6) *is a commutative diagram of bimonoidal 2-categories and 2-strong bimonoidal 2-functors. Moreover,* U_1 *and* U_2 *are strict.*

Chapter 7

Pseudo-Distributive Laws II

In this chapter we investigate pseudo-distributive laws and liftings where the pseudo-endofunctor H has the structure of a pseudo-monad. A pseudo-distributive law over a pseudo-monad is defined to be one over a pseudo-endofunctor that is compatible with the extra structure which makes the pseudo-endofunctor a pseudo-monad. Consequently, a pseudo-distributive law over a pseudo-monad consists of two extra 2-cells for the multiplication and the unit of the second pseudo-monad, and seven extra coherence axioms involving them and the rest of the data, in addition to those three for the pseudo-distributive laws over a pseudo-endofunctor.

We also define a lifting of a pseudo-monad T on a 2-category $\mathbb{C}$ to a pseudo-monad on *Ps-S-Alg*: recall how liftings of pseudo-natural transformations and modifications are defined in Chapter 5. For a pseudo-monad to lift to a pseudo-monad, we require that not only the pseudo-endofunctor itself but also all other components of the pseudo-monad, i.e., the two pseudo-natural transformations and three invertible modifications, lift.

Section 7.3 is the pseudo-version of Section 3.7.2. The definitions given in Section 7.1 and Section 7.2 define the 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\text{ps-monad}}$ of pseudo-distributive laws over pseudo-monads and the 2-category of $\mathsf{Lift}^{\text{ps-monads}}_{\mathit{Ps\text{-}S\text{-}Alg}}$ of liftings to pseudo-monads. We show that these two are equivalent by combining the results of the previous two chapters. We prove that the 2-functors Φ and Ψ we constructed to prove the equivalence between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$ in Section 5.5 naturally define an equivalence between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\text{ps-monad}}$ and $\mathsf{Lift}^{\text{ps-monads}}_{\mathit{Ps\text{-}S\text{-}Alg}}$. The proof of this relies on the result in

Chapter 6 about the bimonoidal structures on both 2-categories.

In Section 7.4 we investigate the property of the composite pseudo-functor TS under the existence of a pseudo-distributive law $ST \to TS$; this is the pseudo-version of Section 3.8 and, similarly to the discussion there on ordinary functors, the pseudo-functor TS has the structure of a pseudo-monad induced by the pseudo-distributive law.

7.1 Pseudo-distributive laws over pseudo-monads

A pseudo-distributive law of a pseudo-monad over a pseudo-monad is similar to a distributive law of an ordinary monad over an ordinary monad. And just as the latter is given by combining distributive laws of a monad over an endofunctor and over a natural transformation, the former is given by combining pseudo-distributive laws of a pseudo-monad over a pseudo-endofunctors, pseudo-natural transformations and modifications, except that in the pseudo-case the commutative axioms are replaced by invertible 2-cells, together with a number of coherence axioms that they need to satisfy.

Definition 7.1. Given pseudo-monads $(S,\mu^S,\eta^S,\tau^S,\lambda^S,\rho^S)$ and $(T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$ on a 2-category $\mathbb{C}$, a *pseudo-distributive law* $(\delta,\overline{\mu}^S,\overline{\mu}^T,\overline{\eta}^S,\overline{\eta}^T)$ of S over T consists of

- a pseudo-natural transformation $\delta : ST \to TS$,
- invertible modifications $\overline{\mu}^S$ and $\overline{\eta}^S$,

$$\begin{tikzcd} S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"'] & STS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}^S"] & TS^2 \arrow[d, "T\mu^S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} T \arrow[d, "\eta^S T"'] \arrow[dr, "T\eta^S"] \arrow[dr, phantom, shift right=3ex, "\Uparrow \overline{\eta}^S"] & \\ ST \arrow[r, "\delta"'] & TS \end{tikzcd}$$

- invertible modifications $\overline{\mu}^T$ and $\overline{\eta}^T$.

$$\begin{tikzcd} ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"'] & STS \arrow[r, "T\delta"] \arrow[d, phantom, "\Downarrow \overline{\mu}^T"] & T^2S \arrow[d, "\mu^T S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} S \arrow[d, "S\eta^T"'] \arrow[dr, "\eta^T S"] \arrow[dr, phantom, shift right=3ex, "\Uparrow \overline{\eta}^T"] & \\ ST \arrow[r, "\delta"'] & TS \end{tikzcd} \tag{7.1}$$

subject to the ten coherence axioms as below. In [Mar99], Marmolejo gave nine coherent axioms for a pseudo-distributive law. However, they seem not to be complete, as some dual axioms are missing. Moreover, the presentation of diagrams in that paper makes calculation difficult, in particular in checking what axioms follow from them. Here we give a more structured presentation, including the apparently missing axiom.

The first axiom involves $\overline{\eta}^S$ and $\overline{\eta}^T$, and is self-dual. This is the same as the first axiom in [Mar99].

$(T\text{-}1)$

$$\begin{tikzcd}
Id_{\mathbb{C}} \arrow[r, "\eta^S"] \arrow[dr, "\eta^T"'] & S \arrow[r, "S\eta^T"] \arrow[dr, "\eta^T S"'] \arrow[d, phantom, "\cong"] & ST \arrow[d, "\delta"] \arrow[dl, phantom, "\Downarrow\overline{\eta}^T"] \\
 & T \arrow[r, "T\eta^S"'] & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Id_{\mathbb{C}} \arrow[r, "\eta^S"] \arrow[dr, "\eta^T"'] & S \arrow[r, "S\eta^T"] \arrow[d, phantom, "\cong"] & ST \arrow[d, "\delta"] \\
 & T \arrow[r, "T\eta^S"'] \arrow[ur, "\eta^S T"] & TS \arrow[ul, phantom, "\Downarrow\overline{\eta}^S"]
\end{tikzcd}$$

Axiom $(T\text{-}2)$ and $(T\text{-}3)$ are coherence between η^S/μ^S and λ^S or ρ^S. $(T\text{-}2)$ is equivalent to (coh 2) in [Mar99], but $(T\text{-}3)$ is missing there.

$(T\text{-}2)$

$$\begin{tikzcd}
 & S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"] & STS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & TS^2 \arrow[d, "T\mu^S"] \\
ST \arrow[ur, "S\eta^S T"] \arrow[r, "id_{ST}"'] \arrow[ur, phantom, "\Downarrow\lambda^S T"'] & ST \arrow[rr, "\delta"'] & & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S^2T \arrow[r, "S\delta"] & STS \arrow[r, "\delta S"] \arrow[d, phantom, "\cong"] & TS^2 \arrow[dr, "T\mu^S"] & \\
ST \arrow[u, "S\eta^S T"] \arrow[ur, "ST\eta^S"'] \arrow[u, phantom, "\Downarrow S\overline{\eta}^S"'] \arrow[rr, "\delta"'] & & TS \arrow[u, "TS\eta^S"] \arrow[r, "id_{TS}"'] \arrow[ur, phantom, "\Downarrow T\lambda^S"] & TS
\end{tikzcd}$$

$(T\text{-}3)$

$$\begin{tikzcd}
 & S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"] & STS \arrow[r, "\delta S"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & TS^2 \arrow[d, "T\mu^S"] \\
ST \arrow[ur, "\eta^S ST"] \arrow[r, "id_{ST}"'] \arrow[ur, phantom, "\Downarrow\rho^S T"'] & ST \arrow[rr, "\delta"'] & & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
S^2T \arrow[r, "S\delta"] & STS \arrow[r, "\delta S"] \arrow[dl, phantom, "\cong"] & TS^2 \arrow[dr, "T\mu^S"] \arrow[dl, phantom, "\Uparrow\overline{\eta}^S S"] & \\
ST \arrow[u, "\eta^S ST"] \arrow[rr, "\delta"'] & & TS \arrow[u, "T\eta^S S"] \arrow[ul, "\eta^S TS"] \arrow[r, "id_{TS}"'] \arrow[ur, phantom, "\Downarrow T\rho^S"] & TS
\end{tikzcd}$$

Axiom $(T\text{-}4)$ and $(T\text{-}5)$ are similar to $(T\text{-}2)$ and $(T\text{-}3)$ (dual, in a sense) and they involve η^T/μ^T and λ^T or ρ^T. These are equivalent to the (coh 8) and (coh 7) of [Mar99] respectively.

$(T\text{-}4)$

$$\begin{tikzcd}
 & ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"] & TST \arrow[r, "T\delta"] \arrow[d, phantom, "\Downarrow\overline{\mu}^T"] & T^2S \arrow[d, "\mu^T S"] \\
ST \arrow[ur, "ST\eta^T"] \arrow[r, "id_{ST}"'] \arrow[ur, phantom, "\Downarrow S\lambda^T"'] & ST \arrow[rr, "\delta"'] & & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
ST^2 \arrow[r, "\delta T"] & TST \arrow[r, "T\delta"] \arrow[dl, phantom, "\cong"] & T^2S \arrow[dr, "\mu^T S"] \arrow[dl, phantom, "\Uparrow T\overline{\eta}^T"] & \\
ST \arrow[u, "ST\eta^T"] \arrow[rr, "\delta"'] & & TS \arrow[u, "T\eta^T S"] \arrow[ul, "TS\eta^T"] \arrow[r, "id_{TS}"'] \arrow[ur, phantom, "\Downarrow\lambda^T S"] & TS
\end{tikzcd}$$

(T-5)
$$
\begin{tikzcd}
& ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"] \arrow[dl, phantom, "\Downarrow S\rho^T"] & TST \arrow[r, "T\delta"] \arrow[d, phantom, "\Downarrow \overline{\mu}^T"] & T^2S \arrow[d, "\mu^T S"] \\
ST \arrow[r, "id_{ST}"'] \arrow[ur, "S\eta^T T"] & ST \arrow[rr, "\delta"'] & & TS
\end{tikzcd}
=
\begin{tikzcd}
ST^2 \arrow[r, "\delta T"] \arrow[dr, phantom, "\Downarrow \overline{\eta}^T T"] & TST \arrow[r, "T\delta"] \arrow[d, phantom, "\cong"] & T^2S \arrow[dr, "\mu^T S"] \arrow[d, phantom, "\Downarrow \rho^T S", shift left=4] & \\
ST \arrow[u, "S\eta^T T"] \arrow[ur, "\eta^T ST"] \arrow[rr, "\delta"'] & & TS \arrow[u, "\eta^T TS"] \arrow[r, "id_{TS}"'] & TS
\end{tikzcd}
$$

Axioms (T-6) and (T-7) involve $\overline{\mu}^S, \tau^S$ and $\overline{\mu}^T, \tau^T$ respectively and are dual to each other. These correspond to (coh 4) and (coh 9) in [Mar99].

(T-6)
$$
\begin{tikzcd}
& S^3T \arrow[r, "S^2\delta"] \arrow[dl, "\mu^S ST"'] \arrow[d, "S\mu^S T"] & S^2TS \arrow[r, "S\delta S"] \arrow[d, phantom, "\Downarrow S\overline{\mu}^S"] & STS^2 \arrow[r, "\delta S^2"] \arrow[d, "ST\mu^S"] & TS^3 \arrow[d, "TS\mu^S"] \arrow[dl, phantom, "\cong"] \\
S^2T \arrow[r, phantom, "\Downarrow \tau^S T"] \arrow[dr, "\mu^S T"'] & S^2T \arrow[rr, "S\delta"] \arrow[d, "\mu^S T"] & \arrow[d, phantom, "\Downarrow \overline{\mu}^S"] & STS \arrow[r, "\delta S"] & TS^2 \arrow[d, "T\mu^S"] \\
& ST \arrow[rrr, "\delta"] & & & TS
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
S^3T \arrow[r, "S^2\delta"] \arrow[d, "\mu^S ST"'] \arrow[dr, phantom, "\cong"] & S^2TS \arrow[r, "S\delta S"] \arrow[d, "\mu^S TS"] & STS^2 \arrow[r, "\delta S^2"] \arrow[d, phantom, "\Downarrow \overline{\mu}^S S"] & TS^3 \arrow[d, "T\mu^S S"'] \arrow[dr, "TS\mu^S"] & \\
S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"'] & STS \arrow[rr, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}^S"] & & TS^2 \arrow[r, phantom, "\Downarrow T\tau^S"] \arrow[d, "T\mu^S"'] & TS^2 \arrow[dl, "T\mu^S"] \\
ST \arrow[rrr, "\delta"] & & & TS &
\end{tikzcd}
$$

(T-7)

$$\begin{tikzcd}
ST^3 \arrow[r, "\delta T^2"] \arrow[d, "S\mu^T T"'] & TST^2 \arrow[r, "T\delta T"] \arrow[d, phantom, "\Downarrow \overline{\mu}^T T"] & T^2ST \arrow[r, "T^2\delta"] \arrow[d, "\mu^T ST"'] \arrow[dr, phantom, "\cong"] & T^3S \arrow[d, "\mu^T TS"'] \arrow[dr, "T\mu^T S"] & {} \\
ST^2 \arrow[rr, "\delta T"] \arrow[d, "S\mu^T"'] \arrow[drr, phantom, "\Downarrow \overline{\mu}^T"] & {} & TST \arrow[r, "T\delta"] & T^2S \arrow[d, "\mu^T S"'] \arrow[r, phantom, "\Downarrow \tau^T S"] & T^2S \arrow[dl, "\mu^T S"] \\
ST \arrow[rrr, "\delta"] & & & TS &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
{} & ST^3 \arrow[r, "\delta T^2"] \arrow[dl, "S\mu^T T"'] \arrow[d, "ST\mu^T"'] \arrow[dr, phantom, "\cong"] \arrow[d, phantom, shift right=6, "\Downarrow S\tau^T"'] & TST^2 \arrow[r, "T\delta T"] \arrow[d, "TS\mu^T"] \arrow[drr, phantom, "\Downarrow T\overline{\mu}^T"] & T^2ST \arrow[r, "T^2\delta"] & T^3S \arrow[d, "T\mu^T S"] \\
ST^2 \arrow[dr, "S\mu^T"'] & ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"] \arrow[drrr, phantom, "\Downarrow \overline{\mu}^T"] & TST \arrow[rr, "T\delta"] & {} & T^2S \arrow[d, "\mu^T S"] \\
{} & ST \arrow[rrr, "\delta"] & & & TS
\end{tikzcd}$$

Axioms (T-8) and (T-9) are dual to each other, involving $\overline{\mu}^S, \overline{\eta}^T$ and $\overline{\mu}^T, \overline{\eta}^S$ respectively. These are equivalent to the (coh 3) and (coh 5) of [Mar99] respectively.

(T-8)

$$\begin{tikzcd}
{} & S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"] & STS \arrow[dr, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}^S"] & {} \\
S^2 \arrow[ur, "S^2\eta^T"] \arrow[d, "\mu^S"'] \arrow[r, phantom, "\cong"] & ST \arrow[drr, "\delta"] & {} & TS^2 \arrow[d, "T\mu^S"] \\
S \arrow[ur, "S\eta^T"] \arrow[rrr, "\eta^T S"'] \arrow[r, phantom, "\Downarrow \overline{\eta}^T"] & {} & & TS
\end{tikzcd}
=
\begin{tikzcd}
{} & S^2T \arrow[r, "S\delta"] \arrow[d, phantom, "\Downarrow S\overline{\eta}^T"] & STS \arrow[dr, "\delta S"] \arrow[d, phantom, "\Downarrow \overline{\eta}^T S"] & {} \\
S^2 \arrow[ur, "S^2\eta^T"] \arrow[urr, "S\eta^T S"] \arrow[rrr, "\eta^T S^2"'] \arrow[d, "\mu^S"'] \arrow[drrr, phantom, "\cong"] & {} & {} & TS^2 \arrow[d, "T\mu^S"] \\
S \arrow[rrr, "\eta^T S"'] & & & TS
\end{tikzcd}$$

(T-9)

$$\begin{tikzcd}
{} & ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"] & TST \arrow[dr, "T\delta"] \arrow[d, phantom, "\Downarrow \overline{\mu}^T"] & {} \\
T^2 \arrow[ur, "\eta^S T^2"] \arrow[d, "\mu^T"'] \arrow[r, phantom, "\cong"] & ST \arrow[drr, "\delta"] & {} & T^2S \arrow[d, "\mu^T S"] \\
T \arrow[ur, "\eta^S T"] \arrow[rrr, "T\eta^S"'] \arrow[r, phantom, "\Downarrow \overline{\eta}^S"] & {} & & TS
\end{tikzcd}
=
\begin{tikzcd}
{} & ST^2 \arrow[r, "\delta T"] \arrow[d, phantom, "\Downarrow \overline{\eta}^S T"] & TST \arrow[dr, "T\delta"] \arrow[d, phantom, "\Downarrow T\overline{\eta}^S"] & {} \\
T^2 \arrow[ur, "\eta^S T^2"] \arrow[urr, "T\eta^S T"] \arrow[rrr, "T^2\eta^S"'] \arrow[d, "\mu^T"'] \arrow[drrr, phantom, "\cong"] & {} & {} & T^2S \arrow[d, "\mu^T S"] \\
T \arrow[rrr, "T\eta^S"'] & & & TS
\end{tikzcd}$$

The last axiom involves $\overline{\mu}^S$ and $\overline{\mu}^T$ and is self dual. This is a reformatted version of (coh 6) in [Mar99].

$$
(T\text{-}10)\qquad
\begin{tikzcd}
S^2T^2 \arrow[r, "S\delta T"] \arrow[d, "S^2\mu^T"'] & STST \arrow[r, "ST\delta"] \arrow[d, phantom, "\Downarrow S\overline{\mu}^T"] & ST^2S \arrow[r, "\delta TS"] \arrow[d, "S\mu^T S"] & TSTS \arrow[r, "T\delta S"] \arrow[d, phantom, "\Downarrow \overline{\mu}^T S"] & T^2S^2 \arrow[d, "\mu^T S^2"] \\
S^2T \arrow[rr, "S\delta"'] \arrow[d, "\mu^S T"'] & & STS \arrow[rr, "\delta S"'] \arrow[d, phantom, "\Downarrow \overline{\mu}^S"] & & TS^2 \arrow[d, "T\mu^S"] \\
ST \arrow[rrrr, "\delta"'] & & & & TS
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
& & & ST^2S \arrow[dr, "\delta TS"] \arrow[d, phantom, "\cong"] & & & \\
& S^2T^2 \arrow[r, "S\delta T"] \arrow[dl, "S^2\mu^T"'] \arrow[d, "\mu^S T^2"] & STST \arrow[ur, "ST\delta"] \arrow[r, "\delta ST"] \arrow[d, phantom, "\Downarrow \overline{\mu}^S T"] & TS^2T \arrow[r, "TS\delta"] \arrow[d, "T\mu^S T"] & TSTS \arrow[r, "T\delta S"] \arrow[d, phantom, "\Downarrow T\overline{\mu}^S"] & T^2S^2 \arrow[d, "T^2\mu^S"'] \arrow[dr, "\mu^T S^2"] & \\
S^2T \arrow[r, phantom, "\cong"] \arrow[dr, "\mu^S T"'] & ST^2 \arrow[rr, "\delta T"'] \arrow[d, "S\mu^T"] & & TST \arrow[rr, "T\delta"'] \arrow[d, phantom, "\Downarrow \overline{\mu}^T"] & & T^2S \arrow[r, phantom, "\cong"] \arrow[d, "\mu^T S"'] & TS^2 \arrow[dl, "T\mu^S"] \\
& ST \arrow[rrrr, "\delta"'] & & & & TS &
\end{tikzcd}
$$

The order in which the axioms are listed here is based on the duality, but there are other perspectives on them. The axioms (T-2), (T-3) and (T-6) correspond to the fact that δ is a pseudo-distributive law of S over an pseudo-endofunctor. Also note that some of the axioms imply that the two invertible modifications $\overline{\mu}^T$ and $\overline{\eta}^T$ in (7.1) are pseudo-distributive laws of S over μ^T and η^T, respectively: the axiom $(T\text{-}10)$ is that of (α^*-1) for $\overline{\mu}^T$, and $(T\text{-}9)$ is that of (α^*-2). Similarly, $(T\text{-}8)$ is that of (α^*-1) for $\overline{\eta}^T$, and $(T\text{-}1)$ is that of (α^*-2).

We now define a 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$ as a variant of $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ consisting only of data involving pseudo-monads;

Proposition 7.2. *The following data constitute a 2-category $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$: the 0-cells are pairs (T,δ^T) of a pseudo-monad $T=(T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$ and a pseudo-distributive law $\delta^T=(\delta^T,\overline{\mu}^S,\overline{\mu}^T,\overline{\eta}^S,\overline{\eta}^T)$ of S over the pseudo-monad. Suppressing the 2-cell data, the 1-cells in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$ from (T,δ^T) to $(T',\delta^{T'})$ are those in*

Ps-Dist^S consisting of pseudo-monad morphisms from T to T'. And finally, the 2-cells in Ps-Dist$^S_{\text{ps-monads}}$ are those in Ps-Dist^S that are defined between pseudo-monad morphisms and preserve the pseudo-monad structure (compatible with the 2-cell components of pseudo-monad morphisms.)

7.2 Lifting a pseudo-monad to *Ps-S-Alg*

Just as in the case of ordinary (non-pseudo) monads, a lifting of a pseudo-monad is a lifting of an endofunctor with some extra conditions, ensuring that the components of the pseudo-monad lift to *Ps-S-Alg*.

Definition 7.3. Given pseudo-monads $(S,\mu^S,\eta^S,\tau^S,\lambda^S,\rho^S)$ and $(T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$ on a 2-category $\mathbb{C}$, a *lifting* of T to *Ps-S-Alg* is a pseudo-monad $\widehat{T}=(\widehat{T},\mu^{\widehat{T}},\eta^{\widehat{T}},\tau^{\widehat{T}},\lambda^{\widehat{T}},\rho^{\widehat{T}})$ for which

$$U\widehat{T}=TU; \tag{7.2a}$$

$$U\mu^{\widehat{T}}=\mu^TU,\quad U\eta^{\widehat{T}}=\eta^TU; \tag{7.2b}$$

$$U\tau^{\widehat{T}}=\tau^TU,\quad U\lambda^{\widehat{T}}=\lambda^TU,\quad U\rho^{\widehat{T}}=\rho^TU. \tag{7.2c}$$

hold, where U is the forgetful pseudo-functor from *Ps-S-Alg* to $\mathbb{C}$.

The condition 7.2a means that $\widehat{T}$ is a lifting of T as a pseudo-endofunctor. Hence (5.3a), (5.3b), and (5.3c) hold for $\widehat{T}$. The conditions 7.2b require that $\mu^{\widehat{T}}$ should be a lifting of μ^T to *Ps-S-Alg* with respect to $\widehat{T}^2$ and $\widehat{T}$, and $\eta^{\widehat{T}}$ that of η^T with respect to $Id_{\textit{Ps-S-Alg}}$ and $\widehat{T}$. (Note that $\widehat{T}^2$ is a lifting of T^2 as an endofunctor.) This means that, for each pseudo-algebra $\langle A,a,a_\mu,a_\eta\rangle$, there exist invertible 2-cells

$$\begin{tikzcd} SA \arrow[r, "S\eta^T_A"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{\eta^T_{A,a}}"] & STA \arrow[d, "\widehat{a}^{\widehat{T}}"] \\ A \arrow[r, "\eta^T_A"'] & TA \end{tikzcd} \qquad \begin{tikzcd} ST^2A \arrow[r, "S\mu^T_A"] \arrow[d, "\widehat{a}^{\widehat{T}^2}"'] \arrow[dr, phantom, "\Downarrow\overline{\mu^T_{A,a}}"] & STA \arrow[d, "\widehat{a}^{\widehat{T}}"] \\ T^2A \arrow[r, "\mu^T_A"'] & TA \end{tikzcd}$$

such that the axioms for pseudo-algebra maps hold, and that the pseudo-naturality of μ^T and η^T extend to that of $\mu^{\widehat{T}}$ and $\eta^{\widehat{T}}$.

And the conditions (7.2c) mean that the liftings $\widehat{T}$, $\mu^{\widehat{T}}$ and $\eta^{\widehat{T}}$ are defined so that the modifications $\tau^T, \lambda^T, \rho^T$ also lift to *Ps-S-Alg* and satisfy the axiom (5.6), serving as $\tau^{\widehat{T}}, \lambda^{\widehat{T}}, \rho^{\widehat{T}}$. For the case of τ^T it amounts to saying that the following holds:

$$
\begin{tikzcd}
ST^3A \arrow[r, "ST\mu^T_A"] \arrow[d, "\widehat{a}^{\widehat{T}^3}"'] \arrow[dr, phantom, "\Downarrow \widehat{T}\overline{\mu^T_{A,a}}"] & ST^2A \arrow[r, "S\mu^T_A"] \arrow[d, "\widehat{a}^{\widehat{T}^2}"'] \arrow[dr, phantom, "\Downarrow \overline{\mu^T_{A,a}}"] & STA \arrow[d, "\widehat{a}^{\widehat{T}}"] \\
T^3A \arrow[r, "T\mu^T_A"'] \arrow[dr, "\mu^T_{TA}"'] & T^2A \arrow[r, "\mu^T_A"'] \arrow[d, phantom, "\Downarrow \tau_A"] & TA \\
& T^2A \arrow[ur, "\mu^T_A"'] &
\end{tikzcd}
=
\begin{tikzcd}
& ST^2A \arrow[dr, "S\mu^T_A"] \arrow[d, phantom, "\Downarrow S\tau_A"] & \\
ST^3A \arrow[ur, "ST\mu^T_A"] \arrow[r, "S\mu^T_{TA}"'] \arrow[d, "\widehat{a}^{\widehat{T}^3}"'] \arrow[dr, phantom, "\Downarrow \overline{\mu^T_{TA,\widehat{a}}}"] & ST^2A \arrow[r, "S\mu^T_A"'] \arrow[d, "\widehat{a}^{\widehat{T}^2}"'] \arrow[dr, phantom, "\Downarrow \overline{\mu^T_{A,a}}"] & STA \arrow[d, "\widehat{a}^{\widehat{T}}"] \\
T^3A \arrow[r, "\mu^T_{TA}"'] & T^2A \arrow[r, "\mu^T_A"'] & TA
\end{tikzcd}
$$

Proposition 7.4. *The liftings of pseudo-monads yield a 2-category* $\mathsf{Lift}^{\textsf{ps-monads}}_{\textit{Ps-S-Alg}}$*: the 0-cells are pairs* $(T, \widehat{T})$*, where* T *is a pseudo-monad and* $\widehat{T}$ *is a lifting of* T *as a pseudo-monad. A 1-cell from* $(T, \widehat{T})$ *to* $(T', \widehat{T}')$ *is a pair* $(\alpha, \widehat{\alpha})$ *of a pseudo-monad morphism* $\alpha : T \to T'$ *and its lifting* $\widehat{\alpha}$ *from* $\widehat{T}$ *to* $\widehat{T}'$*. And a 2-cell from* $(\alpha, \widehat{\alpha})$ *to* $(\beta, \widehat{\beta})$ *is a modification of pseudo-monad morphisms that lifts from* $\widehat{\alpha}$ *to* $\widehat{\beta}$*.*

7.3 Equivalence for the pseudo-monad case

In Section 5.5 we proved that the 2-functors Φ and Ψ define an equivalence between the 2-categories $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{\textit{Ps-S-Alg}}$. In this section, we prove that the 2-functors Φ and Ψ naturally induce 2-functors between $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\textsf{ps-monads}}$ and $\mathsf{Lift}^{\textsf{ps-monads}}_{\textit{Ps-S-Alg}}$, again defining an equivalence between those 2-categories.

Proposition 7.5. *The pseudo-functor* Φ *in Proposition 5.18 induces a pseudo-functor from* $\mathsf{Lift}^{\textsf{ps-monads}}_{\textit{Ps-S-Alg}}$ *to* $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\textsf{ps-monads}}$*.*

Proof. We show that Φ is defined in such a way that it preserves the pseudo-monad structure, hence is a pseudo-functor from $\mathsf{Lift}^{\textsf{ps-monads}}_{\textit{Ps-S-Alg}}$ to $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\textsf{ps-monads}}$.

For 0-cells, given a 0-cell $((T, \mu^T, \eta^T), \widehat{T})$ of $\mathsf{Lift}^{\textsf{ps-monads}}_{\textit{Ps-S-Alg}}$, we first need to show $\Phi((T, \mu^T, \eta^T), \widehat{T}) = ((T, \mu^T, \eta^T), \Phi(\widehat{T}))$ is a 0-cell in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\textsf{ps-monads}}$, i.e., $\Phi(\widehat{T})$ is a pseudo-distributive law over a pseudo-monad rather than a pseudo-endofunctor. What we need to do is to define canonically the extra data (7.1) for pseudo-distributive laws for monads, that is, invertible modifications $\overline{\mu}^T$ and $\overline{\eta}^T$. Then we need to verify those modifications satisfy the coherence axioms. Since $(\mu^T, \mu^{\widehat{T}}) : \widehat{T}^2 \to \widehat{T}$ is a 1-cell in $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$, there exists a pseudo-distributive law $\Phi(\mu^{\widehat{T}}) : \Phi(\widehat{T}^2) \to \Phi(\widehat{T})$ over μ^T. Meanwhile, the equality $\Phi(\widehat{T}^2) = \Phi(\widehat{T} \otimes \widehat{T})$ holds by definition of $\otimes$, and, from Proposition 6.13, we also have

$$\Phi(\widehat{T} \otimes \widehat{T}) \cong \Phi(\widehat{T}) \otimes \Phi(\widehat{T}).$$

Again by definition of $\otimes$ in $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ we have

$$\Phi(\widehat{T}) \otimes \Phi(\widehat{T}) = T\Phi(\widehat{T}) \circ \Phi(\widehat{T})T,$$

which altogether means that there exists a pseudo-distributive law

$$\begin{tikzcd}
ST^2 \arrow[r, "\Phi(\widehat{T})T"] \arrow[d, "S\mu^T"'] & TST \arrow[r, "T\Phi(\widehat{T})"] \arrow[d, phantom, "\Downarrow"] & T^2S \arrow[d, "\mu^T S"] \\
ST \arrow[rr, "\Phi(\widehat{T})"'] & & TS
\end{tikzcd}$$

over μ^T. We define this invertible 2-cell to be $\overline{\mu}^T$ for $\Phi(\widehat{T})$. On the other hand, for $\overline{\eta}^T$, the discussion is slightly simpler than the above; we only need to consider the fact that $(\eta^T, \eta^{\widehat{T}}) : Id_{\mathit{Ps\text{-}S\text{-}Alg}} \to \widehat{T}$ is a 1-cell in $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$. This implies that $\Phi(\eta^{\widehat{T}}) : \Phi(Id) \to \Phi(\widehat{T})$ is a pseudo-distributive law over η^T:

$$\begin{tikzcd}
S \arrow[r, "id_S = \Phi(id_{\mathit{Ps\text{-}S\text{-}Alg}})"] \arrow[d, "S\eta^T_A"'] & S \arrow[d, "\eta^T_{SA}"] \\
ST \arrow[r, "\Phi(\widehat{T})"'] & TS
\end{tikzcd}$$

which provides the invertible 2-cell we define to be $\overline{\eta}^T$.

For the 1-cells and the 2-cells, it is immediate from the definition of Φ in Proposition 5.18 that it preserves pseudo-monad morphisms and modifications between them. Therefore, Φ is a pseudo-functor from $\mathsf{Lift}^{\mathsf{ps\text{-}monads}}_{\mathit{Ps\text{-}S\text{-}Alg}}$ to $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$. $\square$

Proposition 7.6. *The pseudo-functor* Ψ *in Proposition 5.22 induces a pseudo-functor from* $\mathsf{Ps\text{-}Dist}^{S}_{\text{ps-monads}}$ *to* $\mathsf{Lift}^{\text{ps-monads}}_{Ps\text{-}S\text{-}Alg}$.

Proof. Given a 0-cell $((T,\mu^T,\eta^T),\delta^T)$ in $\mathsf{Ps\text{-}Dist}^{S}_{\text{ps-monads}}$, we consider the value

$$\Psi((T,\mu^T,\eta^T),\delta^T) = ((T,\mu^T,\eta^T),\Psi(\delta^T)).$$

We claim that the value is a 0-cell in $\mathsf{Lift}^{\text{ps-monads}}_{Ps\text{-}S\text{-}Alg}$, i.e., the lifting $\Psi(\delta^T)$ of T is that of a pseudo-monad. In order to prove this claim, we need to show that, for the pseudo-monad $T = (T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$, there exist canonical liftings of pseudo-natural transformations μ^T and η^T, and the modifications τ^T,λ^T and ρ^T lift to *Ps-S-Alg*, all from suitable domains to codomains generated by $\Psi(\delta^T)$. For the pseudo-natural transformations μ^T and η^T, since the invertible modifications $\overline{\mu}^T$ and $\overline{\eta}^T$ in (7.1) are pseudo-distributive laws over μ^T and η^T, the values $\Psi(\mu^T,\overline{\mu}^T)$ and $\Psi(\eta^T,\overline{\eta}^T)$ are, by definition of Ψ, liftings of μ^T and η^T, as required. And, for the modifications τ^T, λ^T and ρ^T, the fact that S distribute over them with respect to suitable pseudo-distributive laws over suitable pseudo-natural transformations means that they also lift to *Ps-S-Alg*. Therefore $\Psi(\delta^T)$ is a lifting of a pseudo-monad.

Finally, for the 1-cells and the 2-cells, it is immediate from the definition of Ψ in Proposition 5.22 that it preserves pseudo-monad morphisms and modifications between them. Therefore, Ψ is a pseudo-functor from $\mathsf{Ps\text{-}Dist}^{S}_{\text{ps-monads}}$ to $\mathsf{Lift}^{\text{ps-monads}}_{Ps\text{-}S\text{-}Alg}$. □

Corollary 7.7. *The 2-categories* $\mathsf{Lift}^{\text{ps-monads}}_{Ps\text{-}S\text{-}Alg}$ *and* $\mathsf{Ps\text{-}Dist}^{S}_{\text{ps-monads}}$ *are equivalent.*

7.4 Composite pseudo-monad *TS*

We now consider pseudo-distributive laws and the composite pseudo-monads they induce. Such composite pseudo-monads play a central rôle in the discussion of substitution monoidal structure given in the next chapter.

This section is the pseudo-version of Section 3.8. Given two pseudo-monads S and T and a pseudo-distributive law $ST \to TS$, we show that the composite pseudo-functor TS has the structure of a pseudo-monad. We give a proof to this, which is the pseudo-version of Proposition 3.30. The rest of the results in Theorem 3.36 also extend to the pseudo-setting: we state the corresponding theorem as Theorem 7.9 at the end of the section.

Proposition 7.8. *Given pseudo-monads $(S,\mu^S,\eta^S,\tau^S,\lambda^S,\rho^S)$ and $(T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$ on a 2-category $\mathbb{C}$, and a pseudo-distributive law $\delta : ST \to TS$, the composite pseudo-functor TS acquires the structure for a pseudo-monad on $\mathbb{C}$, with multiplication given by*

$$TSTS \xrightarrow{T\delta} TTSS \xrightarrow{\mu^T\mu^S} TS$$

Proof. The composite of pseudo-functors is a pseudo-functor. Then we need to construct the remaining data for a pseudo-monad $(TS,\mu^{TS},\eta^{TS},\tau^{TS},\lambda^{TS},\rho^{TS})$.

We define the multiplication μ^{TS} and the unit η^{TS} as the following pseudo-natural transformations:

$$\mu^{TS} : TSTS \xrightarrow{T\delta S} TTSS \xrightarrow{TT\mu^S} TTS \xrightarrow{\mu^T S} TS$$
$$\eta^{TS} : Id_{\mathbb{C}} \xrightarrow{\eta^S} S \xrightarrow{\eta^T S} TS$$

Next we define τ^{TS}as in the following diagram:

$$\begin{tikzcd}
TSTSTS \arrow[r, "TST\delta S"] \arrow[d, "T\delta STS"'] \arrow[dr, phantom, "\Downarrow T\delta_{\delta S}"] & TST^2S^2 \arrow[r, "TST^2\mu^S"] \arrow[d, "T\delta TS^2"] \arrow[dr, phantom, "\Downarrow T\delta_{T\mu^S}"] & TST^2S \arrow[r, "TS\mu^T S"] \arrow[d, "T\delta TS"] & TSTS \arrow[dd, "T\delta S"] \\
T^2S^2TS \arrow[r, "T^2S\delta S"'] \arrow[dd, "T^2\mu^S TS"'] \arrow[ddr, phantom, "\Downarrow T^2\overline{\mu}^S S"] & T^2STS^2 \arrow[r, "T^2ST\mu^S"'] \arrow[d, "T^2\delta S^2"] \arrow[dr, phantom, "\Downarrow T^2\delta_{\mu^S}"] & T^2STS \arrow[d, "T^2\delta S"] \arrow[r, phantom, "\Downarrow T\overline{\mu}^{t-1}S"] & \\
 & T^3S^3 \arrow[r, "T^3S\mu^S"'] \arrow[d, "T^3\mu^S S"] \arrow[dr, phantom, "\Downarrow T^3\tau^S"] & T^3S^2 \arrow[r, "T\mu^T S^2"'] \arrow[d, "T^3\mu^S"] \arrow[dr, phantom, "\Downarrow T\mu^T_{\mu^S}"] & T^2S^2 \arrow[d, "T^2\mu^S"] \\
T^2STS \arrow[r, "T^2\delta S"'] \arrow[d, "\mu^T STS"'] \arrow[dr, phantom, "\Downarrow \mu^T_{\delta S}"] & T^3S^2 \arrow[r, "T^3\mu^S"'] \arrow[d, "\mu^T TS^2"] \arrow[dr, phantom, "\Downarrow \mu^T_{T\mu^S}"] & T^3S \arrow[r, "T\mu^T S"'] \arrow[d, "\mu^T TS"] \arrow[dr, phantom, "\Downarrow \tau^T S"] & T^2S \arrow[d, "\mu^T S"] \\
TSTS \arrow[r, "T\delta S"'] & T^2S^2 \arrow[r, "T^2\mu^S"'] & T^2S \arrow[r, "\mu^T S"'] & TS
\end{tikzcd}$$

The left unit λ^{TS} and the right unit ρ^{TS} are defined as: for λ^{TS}

$$
\begin{tikzcd}
TS \arrow[r, "TS\eta^S"] \arrow[ddr, "id_{TS}"'] & TS^2 \arrow[r, "TS\eta^T S"] \arrow[dd, "T\mu^S"] \arrow[dr, "T\eta^T S^2"'] \arrow[ddl, phantom, "\Downarrow T\lambda^S"] & TSTS \arrow[d, "T\delta S"] \arrow[dl, phantom, "\Downarrow T\overline{\eta}^T S"] \\
 & & T^2S^2 \arrow[d, "T^2\mu^S"] \arrow[dl, phantom, "\Downarrow T\eta^T_{\mu^S}"] \\
 & TS \arrow[r, "T\eta^T S"] \arrow[dr, "id_{TS}"'] & T^2S \arrow[d, "\mu^T S"] \arrow[d, phantom, "\Downarrow \lambda^T S", shift right=3ex] \\
 & & TS
\end{tikzcd}
$$

and ρ^{TS}:

$$
\begin{tikzcd}
TS \arrow[r, "\eta^S TS"] \arrow[dr, "T\eta^S S"] \arrow[ddr, "id_{TS}"'] & STS \arrow[r, "\eta^T STS"] \arrow[d, "\delta S"] \arrow[dl, phantom, "\Downarrow \overline{\eta^S} S"] & TSTS \arrow[d, "T\delta S"] \arrow[dl, phantom, "\Downarrow \eta^T_{\delta S}"] \\
 & TS^2 \arrow[r, "\eta^T TS^2"] \arrow[d, "T\mu^S"] \arrow[dl, phantom, "\Downarrow T\rho^S"] & T^2S^2 \arrow[d, "T^2\mu^S"] \arrow[dl, phantom, "\Downarrow \eta^T_{T\mu^S}"] \\
 & TS \arrow[r, "\eta^T TS"] \arrow[dr, "id_{TS}"'] & T^2S \arrow[d, "\mu^T S"] \arrow[d, phantom, "\Downarrow \rho^T S", shift right=3ex] \\
 & & TS
\end{tikzcd}
$$

It is routine to verify that these definitions satisfy the coherence axioms for pseudo-monads shown in Section 2.3. □

The proposition above is the pseudo-version of Proposition 3.30. The rest of the results in Theorem 3.36 also extend to the pseudo-setting:

Theorem 7.9. *Given pseudo-monads $(S,\mu^S,\eta^S,\tau^S,\lambda^S,\rho^S)$ and $(T,\mu^T,\eta^T,\tau^T,\lambda^T,\rho^T)$ on a 2-category Cat, and a pseudo-distributive law $\delta : ST \to TS$,*

- *the composite pseudo-functor TS acquires the structure for a pseudo-monad.*
- *Ps-TS-Alg is canonically equivalent to Ps-$\Psi(\delta)$-Alg*
- *the object $TS1$ has both canonical pseudo-S-algebra and pseudo-T-algebra structures on it.*

Chapter 8

An Application : Substitution Monoidal Structure

In this chapter, we study the intended main application of the theoretical development in this thesis. Every pseudo-distributive laws yields a composite pseudo-monad TS, and hence, a monoidal structure on $TS1$ if T and S are pseudo-monads on *Cat*. We demonstrate that the models of substitution described in [FPT99] and [Tan00] are examples of the monoidal structures thereby induced by pseudo-distributive laws.

We start by looking at examples of pseudo-monads and their pseudo-algebras that interest us in Section 8.1. These pseudo-monads are of two types: the first (Example 8.1, 8.2 and others) is the pseudo-monads on *Cat* that yield the structures for modeling contexts, such as T_{fp} for finite product structure or T_{sm} for symmetric monoidal structure. The second type is the (partial) pseudo-monad on *Cat* for free cocompletion (Example 8.7). We also address the size issue related to this pseudo-monad briefly. In particular, given a strongly inaccessible (or even just regular) cardinals κ, one can consider the free cocompletion under colimits of size less then κ, and that technically may be used to address the size issues.

Section 8.2 gives examples of pseudo-distributive laws between the pseudo-monads described in Section 8.1. We explain why there exist pseudo-distributive laws of each of the pseudo-monads for contexts over the pseudo-monad of free cocompletion, based on the result in Chapter 7 on liftings and that in [IK86].

In the following three sections, we show that an arbitrary pseudo-monad T on *Cat* yields a canonical monoidal structure on the category $T1$. The significance of that monoidal structure is that when T is the pseudo-monad $T_{coc}T_{fp}$, it yields precisely Fiore *et al.*'s substitution monoidal stucture, and likewise for Tanaka when T is $T_{coc}T_{sm}$. Moreover, at the level of generality proposed here, we can follow the main line of development of both pieces of work. The monoidal structure on $T1$ we obtain in this way is the central result here.

We start again from a general discussion for the ordinary (non-pseudo) case in Section 8.3 : given a monoidal category $(\mathcal{C}, \otimes, 1)$ (to recall the definition, see Chapter 2) and a monad (T, μ, η) on it, we give the definition of strength of T. We also show that, given a strong monad T, the object $T1$ in $\mathcal{C}$ has a canonical monoid structure induced by the strength.

This discussion extends naturally to the pseudo-setting: first, in Section 8.4, we define the notion of pseudo-strength of pseudo-monads: we list ten coherence axioms, which bear quite a lot of similarity to those of pseudo-distributive laws, although, for pseudo-strength, it may be possible that one of the ten axioms (axiom (t-9)) is actually redundant. In Section 8.5 we then show that, given a pseudo-monad T with a pseudo-strength on the 2-category *Cat*, the category $T1$, where 1 is the terminal object in *Cat*, has a canonical monoidal structure. This monoidal structure is the main structure for modelling substitution in our examples.

The last section (Section 8.6) then combines the above discussion with that on pseudo-distributive laws from Chapter 7 and demonstrates the substitution monoidal structures for the pseudo-distributive laws given as examples in Section 8.2. We have seen that, given a pseudo-distributive law $\delta : ST \to TS$, the composite TS has the structure of a pseudo-monad (Section 7.4). The monoidal structure $TS1$ induced by this composite pseudo-monad is what we use to model substitution. Those of the pseudo-monads T_{fp} and T_{sm} are precisely the structures of Fiore *et al.* [FPT99] and Tanaka [Tan00]. This chapter is mainly based on the papers [PT04a, PT04b].

8.1 Examples : Pseudo-monads and pseudo-algebras

In this section we give several examples of pseudo-monads on *Cat* and the categories of their pseudo-algebras.

Example 8.1. Let T_{fp} denote the pseudo-monad on *Cat* for small categories with finite products. The 2-category $\mathit{Ps}\text{-}T_{fp}\text{-}\mathit{Alg}$ has objects given by small categories with finite products, arrows given by functors that preserve finite products up to coherent isomorphism (non-strict), and 2-cells by all natural transformations. This is the well-studied 2-category FP of all small categories with finite products. For any small category $\mathcal{C}$, $T_{fp}(\mathcal{C})$ is given by the (bi)free object of FP, i.e., the free category with finite products, on $\mathcal{C}$. Taking $\mathcal{C} = 1$, the category $T_{fp}1$ is given, up to equivalence, by Set_f^{op}, which is denoted by $\mathbb{F}^{op}$ by Fiore *et al*. [FPT99].

Example 8.2. Let T_{sm} denote the pseudo-monad on *Cat* for small symmetric monoidal categories. The 2-category $\mathit{Ps}\text{-}T_{sm}\text{-}\mathit{Alg}$ has objects given by small symmetric monoidal categories, arrows given by strong symmetric monoidal functors, i.e., functors together with data and axioms that makes them preserve the symmetric monoidal structure up to coherent isomorphism, and 2-cells by all symmetric monoidal natural transformations, i.e., those natural transformations that respect the symmetric monoidal structure. This is the well-studied 2-category *SM* of small symmetric monoidal categories. Again, $T_{sm}(\mathcal{C})$ is the (bi)free object of *SM*, i.e, the free symmetric monoidal category on $\mathcal{C}$. Taking $\mathcal{C} = 1$, it follows, up to equivalence, that $T_{sm}(X)$ is the category $\mathbb{P}^{op}$ of finite sets and permutations used by Tanaka [Tan00].

Example 8.3. Lying between the above two examples is the pseudo-monad T_{sm1} on *Cat* for small symmetric monoidal categories whose unit is the terminal object. The 2-category $\mathit{Ps}\text{-}T_{sm1}\text{-}\mathit{Alg}$ has objects given by small symmetric monoidal categories whose unit is the terminal object, arrows given by strong symmetric monoidal functors, and 2-cells by all symmetric monoidal natural transformations. Taking $\mathcal{C} = 1$, it follows that $T_{sm1}(1)$ is given by Inj^{op}, where Inj denotes the category of finite sets and injections. This category has been used by O'Hearn and Tennent, among others, to model local state [OT97].

Example 8.4. Combining the first two examples, specifically by taking the sum of pseudo-monads, we may consider the pseudo-monad T_{BI} defined on *Cat* for small symmetric monoidal categories with finite products. The 2-category *Ps-*T_{BI}*-Alg* has objects given by small symmetric monoidal categories with finite products, arrows given by strong symmetric monoidal functors that preserve finite products, and 2-cells by all symmetric monoidal natural transformations. This structure is used in the Logic of Bunched Implication [Pym02]. The objects of $T_{BI}(\mathcal{C})$ where $\mathcal{C} = 1$ are precisely the *bunches* of Bunched Implications.

The above examples of pseudo-monads allow us to model variable manipulation for untyped variable binding, which will play the rôle of S in a pseudo-distributive law $ST \to TS$ later. We now turn to a (partial) pseudo-monad for cocomplete categories, which will play the rôle of T.

The notion of free cocompletion is defined as follows:

Definition 8.5 (free cocompletion). Given a small category $\mathcal{C}$, the free cocompletion of $\mathcal{C}$ consists of a locally small cocomplete category $\widetilde{\mathcal{C}}$ and a functor $J : \mathcal{C} \to \widetilde{\mathcal{C}}$ such that, for any locally small cocomplete category $\mathcal{D}$, the composition with J induces an equivalence of categories from $Cocomp(\widetilde{\mathcal{C}}, \mathcal{D})$ to $[\mathcal{C}, \mathcal{D}]$.

The next Theorem states that we can always construct such a cocompletion for any small $\mathcal{C}$.

Theorem 8.6 ([Kel82]). *For any small category $\mathcal{C}$, the free cocompletion $\widetilde{\mathcal{C}}$ exists and is given by the category $[\mathcal{C}^{op}, Set]$ together with Yoneda embedding $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{op}, Set]$*

For size reasons, there is no interesting monad on *Cat* for cocomplete categories: small cocomplete categories are necessarily preorders, and the construction of the free (locally small) cocomplete category does not have codomain in *Cat*. Therefore, cocomplete categories do not quite yield a monad or pseudo-monad on *Cat*. But there are well-studied techniques to deal with that concern, essentially by applying size constraints carefully. For instance, assume the existence of a strongly inaccessible cardinal κ. and suppose *Set* has cardinality κ. Now let *CAT* be a universe that contains *Set* (and therefore also *Cat*) as an object. Then the 2-category κ-*cocomp* of categories that are small with respect to CAT and are cocomplete for all diagrams of size less than κ is pseudo-monadic over CAT and the pseudo-monad restricts to the above construction on *Cat* regarded as a full sub-2-category of CAT. Therefore, the construction of the free cocompletion by the Yoneda embedding extends to a pseudo-monad on CAT. For further details see for example [AR94].

Example 8.7. From the above discussion we can safely ignore the size concern. Based on that, there is a (partial) pseudo-monad T_{coc} for cocomplete categories. To the extent to which T_{coc} is a pseudo-monad, $Ps\text{-}T_{coc}\text{-}Alg$ is the 2-category of cocomplete categories, colimit preserving functors, and all natural transformations between them. If follows from Theorem 8.6 that for any small category $\mathcal{C}$, the category $T_{coc}(\mathcal{C})$ is given by the presheaf category $[\mathcal{C}^{op}, Set]$. This construction is fundamental to all of Fiore *et al.*, Tanaka, and Pym [FPT99, Pym02, Tan00]. For variable binding, its universal property has not been considered, but it does provide the key to why their various constructions, in particular their substitution monoidal structures, are definitive, and how they relate to their other structures.

8.2 Examples : Pseudo-distributive laws

In Section 8.1, we gave four examples of pseudo-monads for variable manipulation and one for cocomplete categories. If, in each case, we can give a pseudo-distributive law, it would follow from Theorem 7.9 that the combination of each of the first four pseudo-monads with the fifth would yield a composite pseudo-monad. In fact, pseudo-distributive laws do exist for each of these combinations, by the following argument, based on the main result of [IK86]:

Theorem 8.8. *For a small symmetric monoidal category $\mathcal{C}$, the category $[\mathcal{C}^{op}, Set]$ with the convolution symmetric monoidal structure is the free symmetric monoidal cocompletion of $\mathcal{C}$ with unit given by the Yoneda embedding.*

For an arbitrary small symmetric monoidal category $\mathcal{C}$, the convolution symmetric monoidal structure on $[\mathcal{C}^{op}, Set]$ at (X,Y) is given by the coend

$$X \otimes Y = \int^{i,j} X i \times Y j \times \mathcal{C}(-, i \otimes j), \tag{8.1}$$

where, on the right hand side of the equation, $\otimes$ denotes the symmetric monoidal structure of $\mathcal{C}$, and on the left hand side, $\otimes$ denotes the extension to $[\mathcal{C}^{op}, Set]$. The unit for the convolution symmetric monoidal structure is $\mathcal{C}(-, I)$, where I is the unit of the symmetric monoidal structure of $\mathcal{C}$.

Corollary 8.9. *T_{coc} lifts from Cat to Ps-T_{sm}-Alg.*

Proof. To show that T_{coc} lifts, we need to show that for any small symmetric monoidal category $\mathcal{C}$, the category $T_{coc}\mathcal{C}$ has a symmetric monoidal structure, with the unit η and the multiplication μ strong symmetric monoidal functors. But by Theorem 8.8, the category $T_{coc}\mathcal{C} = [\mathcal{C}^{op}, Set]$ with the convolution symmetric monoidal structure is the free symmetric monoidal cocompletion of $\mathcal{C}$. Moreover, the unit $\eta_{\mathcal{C}}$ is given by the Yoneda embedding, which is strong symmetric monoidal. Again by Theorem 8.8, the category $T_{coc}T_{coc}\mathcal{C}$, with the convolution symmetric monoidal structure, is the free symmetric monoidal cocompletion of $T_{coc}\mathcal{C}$, except for the size concern that we are ignoring. So, up to coherent isomorphism, the identity functor on $T_{coc}\mathcal{C}$, which is strong symmetric monoidal, uniquely extends along the Yoneda embedding to a colimit-preserving strong symmetric monoidal functor from $T_{coc}T_{coc}\mathcal{C}$ to $T_{coc}\mathcal{C}$. But the multiplication $\mu_{\mathcal{C}}$ is, up to coherent isomorphism, the unique extension of the identity map along $\eta_{T_{coc}\mathcal{C}}$ to a T_{coc}-structure-preserving functor. So our extension of the identity functor must agree, up to isomorphism, with $\mu_{\mathcal{C}}$, forcing the latter to be strong symmetric monoidal. $\square$

Example 8.10. It is routine to verify that Corollary 8.9 restricts from symmetric monoidal categories to categories with finite products, i.e., from T_{sm} to T_{fp}. By Corollary 7.7, we therefore have a pseudo-distributive law of T_{fp} over T_{coc}. Applying Theorem 7.9 to T_{fp} and T_{coc}, one obtains the pseudo-monad $T_{coc}T_{fp}$ with $T_{coc}T_{fp}(1)$ being equivalent to $[\mathbb{F}, Set]$, which was Fiore *et al.*'s category for variable binding [FPT99]. One can readily check that the symmetric monoidal structure (8.1) is the finite product structure on $[\mathbb{F}, Set]$, which is calculated point-wise. The unit for the finite product structure, i.e., the terminal object, is given by $\mathbb{F}(-, 1)$.

Example 8.11. By Corollary 8.9 and Corollary 7.7, there is a canonical pseudo-distributive law of T_{sm} over T_{coc}. Applying Theorem 7.9 to T_{sm} and T_{coc}, one obtains the pseudo-monad $T_{coc}T_{sm}$ with $T_{coc}T_{sm}(1)$ equivalent to $[\mathbb{P}, Set]$, which was Tanaka's category for linear variable binding. The symmetric monoidal structure on $[\mathbb{P}, Set]$ is given by the tensor (8.1), which may be expressed as follows:

$$X \otimes Y(n) = \coprod_{n=m_1+m_2} Xm_1 \times Ym_2 \times \mathcal{S}_n / \sim$$

where $\mathcal{S}_n$ is the symmetric group on n elements. Note that $\mathcal{S}_n = \mathbb{P}(n,n)$. The equivalence relation is generated by the relation

$$(x, y, \sigma \circ (\sigma_1 + \sigma_2)) \sim ((X\sigma_1)x, (Y\sigma_2)y, \sigma). \tag{8.2}$$

where x and y are elements of Xm_1 and Ym_2, and σ, σ_1 and σ_2 are permutations on n, m_1 and m_2 ($m_1 + m_2 = n$), respectively. Some simple calculations show that $\mathbb{P}(-, 0)$ is both the left and right unit for $\otimes$.

Example 8.12. Applying a similar discussion to T_{sm1} and T_{coc} yields another composite pseudo-monad with $T_{coc}T_{sm1}(1)$ given by $[Inj, Set]$, as used by O'Hearn and Tennent [OT97].

Example 8.13. Applying a similar discussion to T_{BI} and T_{coc} yields a composite pseudo-monad with $T_{coc}T_{BI}(1)$ given by the functor category $[(T_{BI}1)^{op}, Set]$. The combination of T_{BI} and T_{coc} is implicit in the Logic of Bunched Implications; presheaf categories such as $[(T_{BI}1)^{op}, Set]$ appear explicitly there [Pym02].

8.3 Strength and monoid structure

In the following three sections, we study a property of a pseudo-monad with pseudo-strength on Cat that it induces a monoidal structure on $T1$. This monoidal structure is the one used both in [FPT99] and [Tan00] to construct substitution monoidal structures. As we have done so far, we first study the non-pseudo version of this in this section, which we then extend to the pseudo-version in Section 8.4 and Section 8.5.

Definition 8.14. Given a cartesian closed category $(\mathcal{C}, \times, 1)$, a $\mathcal{C}$-*enriched* monad (T, μ, η) on $\mathcal{C}$ is given by a function

$$ObT : Ob\mathcal{C} \longrightarrow Ob\mathcal{C}$$

together with arrows

$$T_{X,Y} : [X,Y] \longrightarrow [TX,TY]$$

in $\mathcal{C}$, where $[X,Y]$ denotes the internal hom, satisfying commutativity of the diagrams:

$$\begin{tikzcd}
{[Y,Z] \times [X,Y]} \arrow[r, "\circ"] \arrow[d, "T_{Y,Z} \times T_{X,Y}"'] & {[X,Z]} \arrow[d, "T_{X,Z}"] \\
{[TY,TZ] \times [TX,TY]} \arrow[r, "\circ"'] & {[TX,TZ]}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\overline{id}"] \arrow[dr, "\overline{id}"'] & {[X,X]} \arrow[d, "T_{X,X}"] \\
& {[TX,TX]}
\end{tikzcd}$$

together with $\mathcal{C}$-natural transformations $\mu : T^2 \to T$ and $\eta : 1 \to T$ satisfying the usual three conditions for a monad.

Definition 8.15. A *strength* for a monad (T, μ, η) on a cartesian closed category $\mathcal{C}$ consists of a natural transformation with components

$$t_{X,Y} : TX \times Y \longrightarrow T(X \times Y)$$

such that the following diagrams commute: suppressing the associativity isomorphisms of $\mathcal{C}$

$$\begin{tikzcd}
TX \times Y \times Z \arrow[r, "t_{X,Y} \times id"] \arrow[dr, "t_{X,Y\times Z}"'] & T(X \times Y) \times Z \arrow[d, "t_{X\times Y,Z}"] \\
& T(X \times Y \times Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[r, "\cong"] \arrow[d, "id"'] & TX \times 1 \arrow[d, "t_{X,1}"] \\
TX \arrow[r, "\cong"] & T(X \times 1)
\end{tikzcd}$$

$$\begin{tikzcd}
T^2X \times Y \arrow[r, "t_{TX,Y}"] \arrow[d, "\mu \times id"'] & T(TX \times Y) \arrow[r, "Tt_{X,Y}"] & T^2(X \times Y) \arrow[d, "\mu"] \\
TX \times Y \arrow[rr, "t_{X,Y}"'] & & T(X \times Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[d, "\eta \times id"'] \arrow[dr, "\eta"] & \\
TX \times Y \arrow[r, "t_{X,Y}"'] & T(X \times Y)
\end{tikzcd}$$

Theorem 8.16. *To give a $\mathcal{C}$-enriched monad on a cartesian closed category $\mathcal{C}$ is equivalent to giving a strong monad on $\mathcal{C}$.*

A *strong monad* is a monad together with a strength. The next theorem states its property whose pseudo-version gives us the structure we need to model substitution.

Theorem 8.17. *Given a strong monad T on a cartesian closed (more generally a monoidal closed) category $(\mathcal{C}, \times, 1)$, the object $T1$ of $\mathcal{C}$ has a canonical monoid structure with multiplication given by*

$$T1 \times T1 \xrightarrow{t_{1,T1}} T(1 \times T1) \xrightarrow{\cong} T^2 1 \xrightarrow{\mu^T} T1$$

and with unit given by

$$\eta_1 : 1 \longrightarrow T1$$

Moreover, the multiplication $\bullet : T1 \times T1 \longrightarrow T1$ is a T-algebra map in its first variable, i.e.,

$$\begin{tikzcd}
T^2 1 \times T1 \arrow[r, "t_{T1,T1}"] \arrow[d, "\mu \times id"'] & T(T1 \times T1) \arrow[r, "T\bullet"] & T^2 1 \arrow[d, "\mu"] \\
T1 \times T1 \arrow[rr, "\bullet"'] & & T1
\end{tikzcd}$$

commutes.

Sketch of proof. We give a construction for the associativity axiom required for $T1$ to be a monoid. The rest follows from a similar calculation. It is verified by the following diagram:

$$\begin{tikzcd}
T1 \times T1 \times T1 \arrow[rr, "T1 \times t_{1,T1}"] \arrow[d, "t_{1,T1} \times T1"'] \arrow[dr, "t_{1,T1 \times T1}"] & \arrow[d, phantom, "(t)"] & T1 \times T(1 \times T1) \arrow[r, "T1 \times Tl"] \arrow[d, "t_{1,T(1 \times T1)}"] & T1 \times T^2 1 \arrow[r, "T1 \times \mu^T"] \arrow[d, "t_{1,T^2 1}"] & T1 \times T1 \arrow[d, "t_{1,T1}"] \\
T(1 \times T1) \times T1 \arrow[r, "t_{1 \times T1, T1}"'] \arrow[d, "Tl_{T1} \times T1"'] & T(1 \times T1 \times T1) \arrow[r, "T(1 \times t_{1,T1})"'] \arrow[d, "T(l_{T1} \times T1)"] & T(1 \times T(1 \times T1)) \arrow[r, "T(1 \times Tl_{T1})"'] \arrow[d, "Tl_{T(1 \times T1)}"] & T(1 \times T^2 1) \arrow[r, "T(1 \times \mu_1^T)"'] \arrow[d, "Tl_{T^2 1}"] & T(1 \times T1) \arrow[d, "Tl_{T1}"] \\
T^2 1 \times T1 \arrow[r, "t_{T1,T1}"'] \arrow[d, "\mu_1^T \times T1"'] & T(T1 \times T1) \arrow[r, "Tt_{1,T1}"'] & T^2(1 \times T1) \arrow[r, "T^2 l_{T1}"'] \arrow[d, "\mu^T_{1 \times T1}"'] & T^3 1 \arrow[r, "T\mu_1^T"'] \arrow[d, "\mu^T_{T1}"'] & T^2 1 \arrow[d, "\mu_1^T"] \\
T1 \times T1 \arrow[rr, "t_{1,T1}"'] & & T(1 \times T1) \arrow[r, "Tl_{T1}"'] & T^2 1 \arrow[r, "\mu_1^T"'] & T1
\end{tikzcd}$$

which commutes, for the triangle at the top and the pentagon at the bottom, by the axioms for the strength, for the square at the right bottom corner, by the associativity axiom for the monad T, and, for the rest of the squares, by the naturality of the natural transformations appearing in the labelling in each square. Note that the second vertical arrow from the left in the middle row of squares labelled as "$T(l_{T1} \times T1)$" is equal to $T(l_{T1\times T1})$ up to the associativity isomorphism, because $\mathcal{C}$ is a monoidal category. $\square$

A 2-monad is a *Cat*-enriched monad. Therefore a 2-monad T on *Cat* has a strength, and induces a monoid structure on $T1$, i.e., $T1$ is a strict monoidal category. However, when we have a pseudo-monad rather than a 2-monad, the situation is more complex. We need the notion of pseudo-strength, which we introduce in the next section, and it is monoidal structure rather than strict one that a pseudo-monad with a pseudo-strength induces on $T1$.

8.4 Pseudo strength

We seek to generalise the situation for ordinary monads to pseudo-monads on *Cat*. So we need the notion of a pseudo-strength of a pseudo-monad. It is not true that a pseudo-strength is equivalent to a notion of pseudo-enrichment, as we shall explain later, but a pseudo-monad on *Cat* does yield a pseudo-strength, which is all we require. And that in turn yields a monoidal structure on $T1$.

Definition 8.18. A *pseudo-strength* for a pseudo-monad $(T,\mu,\eta,\tau,\lambda,\rho)$ on a bimonoidal 2-category $\mathbb{C} = (\mathbb{C},\otimes,I,\alpha,\lambda,\rho)$ consists of a pseudo-natural transformation with components

$$t_{X,Y} : TX \otimes Y \longrightarrow T(X \otimes Y)$$

and four invertible modifications, whose components are given by the diagrams below:

$$\begin{tikzcd}
(TX\otimes Y)\otimes Z \arrow[r, "t_{X,Y}\otimes id_Z"] \arrow[d, "\alpha"'] & T(X\otimes Y)\otimes Z \arrow[r, "t_{X\otimes Y,Z}"] \arrow[d, phantom, "\Downarrow\bar{t}^1"] & T((X\otimes Y)\otimes Z) \arrow[d, "T\alpha"] \\
TX\otimes(Y\otimes Z) \arrow[rr, "t_{X,Y\otimes Z}"'] & & T(X\otimes(Y\otimes Z))
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \arrow[d, "\rho_{TX}"'] \arrow[dr, "T\rho_X"] \arrow[dr, phantom, shift right=3ex, "\Uparrow\bar{t}^2"] & \\
TX\otimes I \arrow[r, "t_{X,I}"'] & T(X\otimes I)
\end{tikzcd}$$

$$\begin{tikzcd}
T^2X\otimes Y \arrow[r, "t_{TX,Y}"] \arrow[d, "\mu\otimes id"'] & T(TX\otimes Y) \arrow[r, "Tt_{X,Y}"] \arrow[d, phantom, "\Downarrow\bar{t}^3"] & T^2(X\otimes Y) \arrow[d, "\mu"] \\
TX\otimes Y \arrow[rr, "t_{X,Y}"'] & & T(X\otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes Y \arrow[d, "\eta\otimes id"'] \arrow[dr, "\eta"] \arrow[dr, phantom, shift right=3ex, "\Uparrow\bar{t}^4"] & \\
TX\otimes Y \arrow[r, "t_{X,Y}"'] & T(X\otimes Y)
\end{tikzcd}$$

subject to the following ten axioms:

The axioms from (t-1) to (t-3) and (t-10) are for the coherence between the above four invertible modifications. These correspond to (T-1), (T-8), (T-9) and (T-10), respectively, of the axioms for pseudo-distributive laws over pseudo-monads.

(t-1)

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\rho_X"'] & TX \arrow[r, "\rho_{TX}"] \arrow[dr, "T\rho_{X\oplus I}"] \arrow[d, phantom, "\cong_\eta"] & TX\otimes I \arrow[d, "t_{X,I}"] \arrow[dl, phantom, "\Downarrow\bar{t}^2"] \\
& X\otimes I \arrow[r, "\eta_{X\otimes I}"'] & T(X\otimes I)
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\rho_X"'] & TX \arrow[r, "\rho_{TX}"] \arrow[d, phantom, "\cong_\rho"] & TX\otimes I \arrow[d, "t_{X,I}"] \\
& X\otimes I \arrow[r, "\eta_{X\otimes I}"'] \arrow[ur, "\eta_X\oplus id_I"] \arrow[ur, phantom, "\Downarrow\bar{t}^4"', shift right=3] & T(X\otimes I)
\end{tikzcd}
$$

(t-2)

$$
\begin{tikzcd}
& T^2X \arrow[dl, "\rho_{T^2X}"'] \arrow[d, "T\rho_{TX}"] \arrow[dr, "T^2\rho_X"] \arrow[dl, phantom, "\overset{\Longrightarrow}{\bar{t}^2_{TX}}", shift left=6] \arrow[dr, phantom, "\overset{\Longrightarrow}{T\bar{t}^2_X}"', shift right=6] & \\
T^2X\otimes I \arrow[r, "t_{TX,I}"'] \arrow[d, "\mu_X\otimes id"'] & T(TX\otimes I) \arrow[r, "Tt_{X,I}"'] \arrow[d, phantom, "\Downarrow\bar{t}^3_{X,I}"] & T^2(X\otimes I) \arrow[d, "\mu_{X\otimes I}"] \\
TX\otimes I \arrow[rr, "t_{X,I}"'] & & T(X\otimes I)
\end{tikzcd}
=
\begin{tikzcd}
& T^2X \arrow[dl, "\rho_{T^2X}"'] \arrow[d, "\mu_X"] \arrow[dr, "T^2\rho_X"] & \\
T^2X\otimes I \arrow[r, phantom, "\cong_\rho"] \arrow[d, "\mu_X\otimes id"'] & TX \arrow[dl, "\rho_{TX}"'] \arrow[dr, "T\rho_X"] \arrow[d, phantom, "\Uparrow\bar{t}^2_X"] \arrow[r, phantom, "\cong_\mu"] & T^2(X\otimes I) \arrow[d, "\mu_{X\otimes I}"] \\
TX\otimes I \arrow[rr, "t_{X,I}"'] & & T(X\otimes I)
\end{tikzcd}
$$

(t-3)

$$
\begin{tikzcd}
& (X\otimes Y)\otimes Z \arrow[dl, "(\eta\oplus id)\oplus id"'] \arrow[d, "\eta\otimes id"] \arrow[dr, "\eta"] \arrow[dl, phantom, "\overset{\Longrightarrow}{\bar{t}^4\otimes id}", shift left=6] \arrow[dr, phantom, "\overset{\Longrightarrow}{T\bar{t}^4}"', shift right=6] & \\
(TX\otimes Y)\otimes Z \arrow[r, "t\otimes id"'] \arrow[d, "\alpha"'] & T(X\otimes Y)\otimes Z \arrow[r, "t"'] \arrow[d, phantom, "\Downarrow\bar{t}^1"] & T((X\otimes Y)\otimes Z) \arrow[d, "\alpha"] \\
TX\otimes(Y\otimes Z) \arrow[rr, "t_{X,Y\otimes Z}"'] & & T(X\otimes(Y\otimes Z))
\end{tikzcd}
=
\begin{tikzcd}
& (X\otimes Y)\otimes Z \arrow[dl, "(\eta\oplus id)\oplus id"'] \arrow[d, "\alpha"] \arrow[dr, "\eta"] & \\
X\otimes(Y\otimes Z) \arrow[r, phantom, "\cong_\alpha"] \arrow[d, "\alpha"'] & (X\otimes Y)\otimes Z \arrow[dl, "\eta\oplus id"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\Downarrow\bar{t}^4"] \arrow[r, phantom, "\cong_\eta"] & T((X\otimes Y)\otimes Z) \arrow[d, "\alpha"] \\
TX\otimes(Y\otimes Z) \arrow[rr, "t_{X,Y\otimes Z}"'] & & T(X\otimes(Y\otimes Z))
\end{tikzcd}
$$

Axioms (t-4) to (t-6) involve the invertible modifications $\bar{t}^3$ and $\bar{t}^4$, which are the data for the pseudo-strength and the multiplication and the unit for the pseudo-monad T. These axioms are for the coherence between these two invertible modifications and those from the data of the pseudo-monad T. (t-4) is the axiom for the left unit λ, (t-5) for the right unit ρ, and (t-6) for the multiplication τ of the pseudo-monad.

(t-4)

$$\begin{tikzcd}
 & T^2X \otimes Y \arrow[r, "t_{TX,Y}"] \arrow[d, "\mu \otimes id_Y"] \arrow[dl, phantom, "\Downarrow \lambda \otimes id"] & T(TX \otimes Y) \arrow[r, "Tt_{X,Y}"] \arrow[d, phantom, "\Downarrow \bar{t}^3"] & T^2(X \otimes Y) \arrow[d, "\mu_{X \otimes Y}"] \\
TX \otimes Y \arrow[ur, "T\eta_X \otimes id_Y"] \arrow[r, "id"'] & TX \otimes Y \arrow[rr, "t_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
T^2X \otimes Y \arrow[r, "t_{TX,Y}"] \arrow[dr, phantom, "\cong_t"] & T(TX \otimes Y) \arrow[r, "Tt_{X,Y}"] \arrow[dr, phantom, "\Uparrow T\bar{t}^4"] & T^2(X \otimes Y) \arrow[dr, "\mu_{X \oplus Y}"] \arrow[dr, phantom, "\Downarrow \lambda_{X \otimes Y}"'] & \\
TX \otimes Y \arrow[u, "T\eta_X \otimes id_Y"] \arrow[rr, "t_{X,Y}"'] & & T(X \otimes Y) \arrow[ul, "T(\eta_X \oplus id)"'] \arrow[u, "T\eta_{X \otimes Y}"'] \arrow[r, "id"'] & T(X \otimes Y)
\end{tikzcd}$$

(t-5)

$$\begin{tikzcd}
 & T^2X \otimes Y \arrow[r, "t"] \arrow[d, "\mu \otimes id_Y"] \arrow[dl, phantom, "\Downarrow \rho \otimes id"] & T(TX \otimes Y) \arrow[r, "Tt"] \arrow[d, phantom, "\Downarrow \bar{t}^3"] & T^2(X \otimes Y) \arrow[d, "\mu_{X \otimes Y}"] \\
TX \otimes Y \arrow[ur, "\eta_{TX} \otimes id_Y"] \arrow[r, "id"'] & TX \otimes Y \arrow[rr, "t_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
T^2X \otimes Y \arrow[r, "t"] & T(TX \otimes Y) \arrow[r, "Tt"] \arrow[dr, phantom, "\cong_\eta"] & T^2(X \otimes Y) \arrow[dr, "\mu_{X \otimes Y}"] \arrow[dr, phantom, "\Downarrow \rho_{X \otimes Y}"'] & \\
TX \otimes Y \arrow[u, "\eta_{TX} \otimes id_Y"] \arrow[ur, "\eta_{TX \otimes Y}"'] \arrow[u, phantom, "\Downarrow \bar{t}^4"'] \arrow[rr, "t_{X,Y}"'] & & T(X \otimes Y) \arrow[u, "\eta_{T(X \otimes Y)}"] \arrow[r, "id"'] & T(X \otimes Y)
\end{tikzcd}$$

(*t*-6)

$$
\begin{tikzcd}
 & T^3X\otimes Y \arrow[r, "t_{T^2X,Y}"] \arrow[dl, "\mu_{TX}\otimes id"'] \arrow[d, "T\mu_X\otimes id"] \arrow[dr, phantom, "\Downarrow t_{\mu_X}"] & T(T^2X\otimes Y) \arrow[r, "Tt_{TX}"] \arrow[d, "T(\mu_X\otimes id)"] & T^2(TX\otimes Y) \arrow[r, "T^2t_{X,Y}"] \arrow[d, phantom, "\Downarrow T\bar{t}^3_{X,Y}"] & T^3(X\otimes Y) \arrow[d, "T\mu_{X\otimes Y}"] \\
T^2X\otimes Y \arrow[r, phantom, "\Downarrow \tau_X\otimes id"] \arrow[dr, "\mu_X\otimes id"'] & T^2X\otimes Y \arrow[r, "t_{TX,Y}"'] \arrow[d, "\mu_X\otimes id"] & T(TX\otimes Y) \arrow[rr, "Tt_{X,Y}"'] \arrow[drr, phantom, "\Downarrow \bar{t}^3_{X,Y}"] & & T^2(X\otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
 & TX\otimes Y \arrow[rrr, "t_{X,Y}"] & & & T(X\otimes Y)
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
T^3X\otimes Y \arrow[r, "t_{T^2X,Y}"] \arrow[d, "\mu_{TX}\otimes id"'] \arrow[drr, phantom, "\Downarrow \bar{t}^3_{TX,Y}"] & T(T^2X\otimes Y) \arrow[r, "Tt_{TX}"] & T^2(TX\otimes Y) \arrow[r, "T^2t_{X,Y}"] \arrow[d, "\mu_{TX\otimes Y}"] \arrow[dr, phantom, "\Downarrow \mu_t"] & T^3(X\otimes Y) \arrow[d, "\mu_{T(X\otimes Y)}"] \arrow[dr, "T\mu_{X\otimes Y}"] & \\
T^2X\otimes Y \arrow[rr, "t_{TX,Y}"] \arrow[d, "\mu_X\otimes id"'] \arrow[drrr, phantom, "\Downarrow \bar{t}^3_{X,Y}"] & & T(TX\otimes Y) \arrow[r, "Tt_{X,Y}"'] & T^2(X\otimes Y) \arrow[r, phantom, "\Downarrow \tau_{X\otimes Y}"] \arrow[d, "\mu_{X\otimes y}"] & T^2(X\otimes Y) \arrow[dl, "\mu_{X\otimes Y}"] \\
TX\otimes Y \arrow[rrr, "t_{X,Y}"] & & & T(X\otimes Y) &
\end{tikzcd}
$$

Axioms (*t*-7), (*t*-8) and (*t*-9) are for the invertible modification $\bar{t}^1$ and $\bar{t}^2$, which involve the isomorphisms α and ρ of the bimonoidal structure on $\mathbb{C}$. The axiom (*t*-7) is for the coherence between $\bar{t}^1$ and the constraint for the associativity isomorphism α, while (*t*-8) is for that between $\bar{t}^1$, $\bar{t}^2$ and the constraint for the left and right unit λ, ρ

isomorphisms.

The axiom (t-9) is for the coherence of the derived constraint for the right unit isomorphism ρ. It is likely, that this is actually redundant, given the fact that the constraint is a derived one, the proof of which should provide the main part of the proof of its redundancy.

(t-8)

$$\begin{tikzcd}
 & (TX\otimes I)\otimes Y \arrow[r, "t_{X,I}\otimes id_Y"] \arrow[d, "\alpha"] & T(X\otimes I)\otimes Y \arrow[r, "t_{X\otimes I,Y}"] \arrow[d, phantom, "\Downarrow \bar{t}^1"] & T((X\otimes I)\otimes Y) \arrow[d, "T\alpha"] \\
TX\otimes Y \arrow[ur, "\rho_{TX}\otimes id_Y"] \arrow[r, "id_{TX}\otimes\lambda_Y"'] \arrow[ur, phantom, "\Downarrow *_{TX,I}"'] & TX\otimes(I\otimes Y) \arrow[rr, "t_{X,I\otimes Y}"'] & & T(X\otimes(I\otimes Y))
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
(TX\otimes I)\otimes Y \arrow[r, "t_{X,I}\otimes id_Y"] \arrow[dr, phantom, "\Downarrow \bar{t}^2\otimes id_Y"] & T(X\otimes I)\otimes Y \arrow[r, "t_{X\otimes I,Y}"] \arrow[d, phantom, "\cong_t"] & T((X\otimes I)\otimes Y) \arrow[dr, "T\alpha"] \arrow[dr, phantom, "\Downarrow *_{TX,I}"'] & \\
TX\otimes Y \arrow[u, "\rho_{TX}\otimes id_Y"] \arrow[ur, "T\rho_X\otimes id_Y"'] \arrow[rr, "t_{X,Y}"'] \arrow[dr, "id_{TX}\otimes\lambda_Y"'] & {} & T(X\otimes Y) \arrow[u, "T(\rho_X\otimes id_Y)"'] \arrow[r, "T(id_X\otimes\lambda_Y)"'] & T(X\otimes(I\otimes Y)) \\
 & TX\otimes(I\otimes Y) \arrow[u, phantom, "\cong_t"] \arrow[urr, "t_{X,I\otimes Y}"'] & &
\end{tikzcd}$$

(t-9)

$$\begin{tikzcd}
 & (TX\otimes Y)\otimes I \arrow[r, "t_{X,Y}\otimes id_I"] \arrow[d, "\alpha"] & T(X\otimes Y)\otimes I \arrow[r, "t_{X\otimes Y,I}"] \arrow[d, phantom, "\Downarrow \bar{t}^1"] & T((X\otimes Y)\otimes I) \arrow[d, "T\alpha"] \\
TX\otimes Y \arrow[ur, "\rho_{TX\otimes Y}"] \arrow[r, "id_{TX}\otimes\rho_Y"'] \arrow[ur, phantom, "\Downarrow *^{\rho}_{TX,Y}"'] & TX\otimes(Y\otimes I) \arrow[rr, "t_{X,Y\otimes I}"'] & & T(X\otimes(Y\otimes I))
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
(TX\otimes Y)\otimes I \arrow[r, "t_{X,Y}\otimes id_I"] \arrow[dr, phantom, "\cong_\rho"] & T(X\otimes Y)\otimes I \arrow[r, "t_{X\otimes Y,I}"] \arrow[dr, phantom, "\overset{\Longrightarrow}{\bar{t}^2}"] & T((X\otimes Y)\otimes I) \arrow[dr, "T\alpha"] \arrow[dr, phantom, "\Downarrow T*^{\rho}_{X,Y}"'] & \\
TX\otimes Y \arrow[u, "\rho_{TX\otimes Y}"] \arrow[rr, "t_{X,Y}"'] \arrow[dr, "id_{TX}\otimes\rho_Y"'] & {} & T(X\otimes Y) \arrow[ul, "\rho_{T(X\otimes Y)}"] \arrow[u, "T\rho_{X\otimes Y}"'] \arrow[r, "T(id_X\otimes\rho_Y)"'] & T(X\otimes(Y\otimes I)) \\
 & TX\otimes(Y\otimes I) \arrow[u, phantom, "\cong_t"] \arrow[urr, "t_{X,Y\otimes I}"'] & &
\end{tikzcd}$$

One can formulate a coherence theorem for pseudo-strengths that shows the definitiveness of the axioms up to provable equality. It is a mild diversion to investigate it here, but we shall state the result for completeness.

Proposition 8.19. *Given a pseudo-strength t for a pseudo-monad T on a 2-category $\mathbb{C}$, and given any parallel pair of pseudo-natural transformations constructed from copies of t, μ, and η, there is a unique modification between them constructed from the modifications in the data for a pseudo-monad and a pseudo-strength.*

Theorem 8.20. *Every pseudo-monad on Cat gives rise to a pseudo-strength.*

Proof. Define $t_{X,Y}$ by Currying

$$Y \longrightarrow [X, X \times Y] \xrightarrow{T} [TX, T(X \times Y)]$$

The rest of the data for pseudo-naturality arises from pseudo-functoriality of T, as do the first two structural modifications. The latter two structural modifications arise from the pseudo-naturality of μ and η. Verification of the axioms is routine. ☐

The reason one does not have a meaningful equivalence between the notions of pseudo-monad and pseudo-strength is because a pseudo-monad may have an underlying pseudo-functor that is not an ordinary functor.

8.5 Monoidal structure on *T1*

Now we state the pseudo-version of Theorem 8.17. We give the theorem without a proof. The verification is lengthy but routine.

Theorem 8.21. *Given a pseudo-monad T on Cat, the category $T1$ has a canonical monoidal structure with composition defined by using the pseudo-strength induced by T as follows:*

$$T1 \times T1 \xrightarrow{t_{1,T1}} T(1 \times T1) \xrightarrow{\cong} T^2 1 \xrightarrow{\mu_1} T1$$

and with unit given by

$$\eta_1 : 1 \longrightarrow T1$$

The associativity and unit isomorphisms are generated by those for the multiplication and unit of T together with those of the pseudo-strength. Moreover, the multiplication $\bullet : T1 \times T1 \longrightarrow T1$ is a pseudo-map of T-algebras in its first variable, i.e., there is a coherent isomorphism

$$\begin{tikzcd}
T^2 1 \times T1 \arrow[r, "t_{T1,T1}"] \arrow[d, "\mu \times id"'] & T(T1 \times T1) \arrow[r, "T\bullet"] \arrow[d, phantom, "\cong"] & T^2 1 \arrow[d, "\mu"] \\
T1 \times T1 \arrow[rr, "\bullet"'] & & T1
\end{tikzcd}$$

8.6 Examples : Substitution monoidal structures

Applying this result to the pseudo-monad TS obtained from Theorem 7.9, $TS1$ is a monoidal category, i.e., a monoid in *Cat*, with the multiplication $\bullet$.

Calculating the value $X \bullet Y$ for objects X and Y of $T1$ is not easy in general, but the final clause of Theorem 8.21 makes life easier. Typically, an object of $T1$ is given by a sophisticated sort of word of copies of 1. But $1 \bullet Y$ must always be isomorphic to Y. So the final clause of the theorem tells us that, if we express X as a word of copies of 1, the object $X \bullet Y$ is given by replacing each copy of 1 in that word by an occurrence of Y. This fact, together with that T is given by $T_{coc}S$ in our cases, enable the tensor $\bullet$ to be readily calculated.

Example 8.22. Consider the pseudo-monad $T_{coc}T_{fp}$ on *Cat*. We have already seen that $T_{coc}T_{fp}(1)$ is equivalent to $[\mathbb{F}, Set]$. So, by the theorem, $[\mathbb{F}, Set]$ acquires a canonical monoidal structure. By the last line of the theorem, for every object X of $[\mathbb{F}, Set]$, the functor $- \bullet Y : [\mathbb{F}, Set] \longrightarrow [\mathbb{F}, Set]$ is a pseudo-map of $T_{coc}T_{fp}$-algebras, and so preserves both colimits and finite products. Since every functor $X : \mathbb{F} \longrightarrow Set$ is a colimit of representables, and every object of $\mathbb{F}^{op}$ is a finite product of copies of the generating object 1, which in turn is the unit of $\bullet$, it follows that we can calculate $X \bullet Y$ as a canonical coequaliser of the form

$$(X \bullet Y)m = \coprod_{n \varepsilon N} (Xn \times (Ym)^n)/\sim$$

yielding exactly Fiore *et al.*'s construction of a substitution monoidal structure.

Example 8.23. Consider the pseudo-monad $T_{coc}T_{sm}$ on *Cat*. We have already seen that $T_{coc}T_{sm}(1)$ is equivalent to $[P,Set]$. Applying the same argument as in the previous example, we can calculate $X \bullet Y$ and check that it agrees with Tanaka's construction of a substitution monoidal structure, namely

$$(X \bullet Y)m = \coprod_{n \epsilon N} (Xn \times (Y^{(n)})m)/\sim$$

where $Y^{(n)}$ denotes the n-fold tensor product in $[\mathbb{P},Set]$ of Y, using the convolution symmetric monoidal product of $[\mathbb{P},Set]$: that convolution symmetric monoidal product is exactly the lifting to $[\mathbb{P},Set]$ of the canonical symmetric monoidal product of $\mathbb{P}^{op}$, which is, in turn, the free symmetric monoidal category on 1, i.e., $T_{sm}(1)$. The reason one still sees a product in this formula is because, conceptually, it plays the role of the Xn-fold sum of copies of $Y^{(n)}m$ here rather than that of a product.

Example 8.24. Considering $T_{coc}T_{BI}$, one can make a similar calculation: every functor $X : (T_{BI})^{op} \to Set$ is a colimit of representables, and each representable is a bunch of copies of 1. So, if one takes a formula for X as a colimit of bunches of copies of 1 and replaces each occurrences of 1 by Y, one obtains a formula for $X \bullet Y$ of the form

$$(X \bullet Y)b = \int^{b' \in T_{BI}1} Xb' \times (Y^{(b')})b$$

where $Y^{(b')}$ represents a b'-bunches of copies of the object Y of $[(T_{BI})^{op},Set]$. This integral may be calculated as

$$(X \bullet Y)b = \coprod_{b' \in T_{BI}1} (Xb' \times (Y^{(b')})b)/\sim$$

for an equivalence relation $\sim$ generated similarly to those of Examples 8.22 and 8.23.

One can also apply the same style of analysis to the other examples, yielding canonical substitution monoidal structures for, for instance, affine binders. It follows in general, from the fact that we always consider T_{coc}, that our generalised substitution monoidal structure is always closed. That closedness appears in Fiore *et al.*'s work, in Tanaka's work, and in the work on Bunched Implications. Moreover, we know from the previous section that, given any pseudo-distributive law, $TS(1)$ always has a pseudo-S-algebra structure. That agrees with the finite product structure of Fiore *et al*. and it agrees with the corresponding symmetric monoidal structure of Tanaka as remarked in the last example above.

Chapter 9

Conclusions and Further work

9.1 Conclusions

In this thesis we have investigated the properties of pseudo-distributive laws of pseudo-monads over pseudo-monads, and, as an application of the investigation, we constructed a framework that provides the structures for modeling substitution for terms in contexts with different structural properties. This is a structure that unifies the category-theoretic formulations of substitution in higher order abstract syntax discussed in Fiore *et al*. [FPT99] and also in [Tan00].

The definition of pseudo-distributive laws of pseudo-monads over pseudo-monads was given together with its ten coherence axioms. These coherence axioms arise from the facts that such pseudo-distributive laws should let each datum of pseudo-monads (pseudo-natural transformations and modifications) naturally be endowed with suitable distributivity. We believe that this definition of pseudo-distributive laws of pseudo-monads over pseudo-monads is definitive in the sense that the axioms are complete and elegant. Moreover, we introduced the notions of pseudo-distributivity generally; definitions of the pseudo-distributivity of a pseudo-monad over pseudo-natural transformations and over modifications were given alongside the pseudo-distributive laws of a pseudo-monad over pseudo-endofunctors, allowing the application of these definitions for combinations of such structures, for instance, the case of comonads, as in the work of Winskel in modelling bisimilarity.

In the analysis of the properties of a pseudo-distributive law of a pseudo-monad S over a pseudo-endofunctor H, we introduced the notion of lifting of H to the 2-category of pseudo-S-algebras. We then provided a proof that the notions of pseudo-distributive laws and of liftings are equivalent in the sense that they define equivalent 2-categories. We also proved, not in the pseudo-setting but in terms of ordinary categories and functors, that, in a sense, a "dual" to this also holds, in that a distributive law of an endofunctor H over a monad T is equivalent to an extension of H to the Kleisli category $Kl(T)$ of T, which is easily extended to the pseudo-case. In moving from pseudo-distributive laws over pseudo-endofunctors to those over pseudo-monads, we investigated the bimonoidal structures on $\mathsf{Ps\text{-}Endo}(\mathbb{C})$, $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}$ and $\mathsf{Lift}_{\mathit{Ps\text{-}S\text{-}Alg}}$, which provide the canonical composition both of liftings and of pseudo-distributive laws. This fact is essential in the proof of equivalence between the 2-categories $\mathsf{Ps\text{-}Dist}^{\mathsf{S}}_{\mathsf{ps\text{-}monads}}$ of pseudo-distributive laws over pseudo-monads and $\mathsf{Lift}^{\mathsf{ps\text{-}monads}}_{\mathit{Ps\text{-}S\text{-}Alg}}$ of liftings of pseudo-monads to pseudo-monads on $\mathit{Ps\text{-}S\text{-}Alg}$. Another important property of pseudo-distributive laws of a pseudo-monad S over a pseudo-monad T is that when there exists such a pseudo-distributive law, the composite pseudo-functor TS acquires the structure of a pseudo-monad. We proved this in Section 7.4.

As the main examples of our analysis we consider two different types of pseudo-monads: the first of them are the pseudo-monads that give categories that are used to model various different types of contexts, such as the pseudo-monad for finite product structure and that for symmetric monoidal structure. The other type of pseudo-monad is that for the free cocompletion, modulo the size issue, which, for a small category $\mathcal{C}$, gives its free cocompletion $[\mathcal{C}, \mathit{Set}]$. We explained why there exist pseudo-distributive laws for combinations of one of the pseudo-monads for contexts and that of free cocompletion, which follows using Im and Kelly's work in [IK86].

Moving back from the examples to the discussion of general structure, we then considered the monoidal structure induced by the notion of pseudo-strength. Similarly to the case of an ordinary strength and a monad, we have the fact that any pseudo-monad T on Cat has a pseudo-strength. From there we showed that there exists a monoidal structure on the object $T1$ in Cat induced by the pseudo-strength. We consider pseudo-monads of the form TS, where T is the cocompletion pseudo-monad and S is one of the pseudo-monads for contexts. The category $TS1$ has the form $[(S1)^{op}, Set]$ and the tensor for the monoidal structure induced by the pseudo-strength for this category is calculated as a coend, due to the fact that T is the free cocompletion monad.

9.2 Further work

There are many possibilities for the future work of this thesis. The most important is the further investigation of the syntactical aspects of the unifying framework. A definition of binding signatures for generic contexts should be given in such a way that the signatures defined in [FPT99] and [Tan00] are included as instances and also the functor that is associated to such a signature should have a strength with respect to the tensor for substitution discussed in Chapter 8.

9.2.1 Syntactic aspects

The definitions of binding signature given in both [FPT99] and [Tan00] are in fact identical:

Definition 9.1 ([FPT99],[Tan00]). A *binding signature* $\Sigma = (O, a)$ consists of a set of *operations* O and an *arity* function $a : O \to \mathbb{N}^*$.

An operator o of arity $\langle n_1, \ldots, n_k\rangle$ has k arguments and binds n_i variables in the i-th argument ($1 \leq i \leq n_i$). The terms associated to a signature Σ over a set of variables ranged over by x are given by the grammar:

$$t \in T_\Sigma ::= x \mid o((x_1, \ldots, x_{n_1}).t_1, \ldots, (x'_1, \ldots, x'_{n_k}).t_k)$$

where o is in O and $a(o) = \langle n_1, \ldots, n_k\rangle$. The notions of free/bound variables and α-equivalence are defined in the obvious way.

In the linear case, each variable to be bound by a binder has exactly one occurrence in the term, where as in the ordinary case there is no such restriction. However, such facts do not surface in the definition of signatures itself. The distinction is introduced when one considers the notion of *binding algebra* on the suitable presheaf category.

In order to interpret an operation o of arity $\langle n_1, \ldots, n_k \rangle$, the algebra on a presheaf X associated to this operation has the forms

$$(\delta^{n_1} X) \times \cdots \times (\delta^{n_k} X) \to X$$

for ordinary binders and

$$(\partial^{n_1} X) \otimes \cdots \otimes (\partial^{n_k} X) \to X$$

for linear binders, where δX and ∂X are defined to be $X(1 + -)$ with the operation $+$ interpreted in $\mathbb{F}$ and $\mathbb{P}$ respectively. Both δ and ∂ are used to give a mathematical formulation of the idea of binding over one variable. The definition for ordinary binders uses the finite product structure of both $\mathbb{F}^{op}$ (finite products in $\mathbb{F}^{op}$ are finite coproducts in $\mathbb{F}$) and $[\mathbb{F}, Set]$, together with the object 1 of $\mathbb{F}$. As we have seen in earlier chapters, that is all elegantly expressible as structure generated by the 2-monad T_{fp}. The same is true for linear binders, in which case the definition $X(1 + -)$ of ∂X is given not by the coproduct but by the symmetric monoidal structure on $\mathbb{P}$. Moreover, the symmetric monoidal structure on $[\mathbb{P}, Set]$ is used instead of the product. This is again a structure generated by the 2-monad T_{sm}.

But this definition of binding signature in [FPT99] and [Tan00], although fine for their purposes, is insufficient in more complex binding situations, where more than two kinds of binders may be present in the signature, for instance, that of Bunched Implications. In Bunched Implications, one has two sorts of binders: a linear binder and a non-linear binder. So we need to be able to specify the kind of binding an operator employs to bind a particular argument. A finite sequence of natural numbers is not precise enough to specify which sort of binder is to be used, and in what combination the binders are to be used.

So, in order to capture such examples in which one has more than one binder, one needs a more refined general notion of binding signature. We do not have a definitive general account of that yet, so we shall not develop that idea here beyond mentioning that it definitely is possible to unify these examples and extend them to situations such as that of Bunched Implications as explained in [Pow03], the only question being how elegantly one can do so and with what degree of definitiveness.

The above definition of binding signature essentially contains two pieces of data: for each i, each n_i tells you how many times to apply $X(1 + -)$, and k tells you how many such $X(n_i + -)$ ($= \delta^{n_i}X$ or $\partial^{n_i}X$) need to be multiplied. More generally, we need to allow more freedom than that, so that, for the cases such as Bunched Implications, we may specify words of products and tensors, with the n_i and the k only telling us how long those words are. In fact, in the general setting of 2-monads on *Cat*, the notion of a Lawvere 2-theory [Pow99] supports such a general notion of signature. We do not go into the details, but it supports the following definition.

Let S denote one of the 2-monads for the structures that model contexts. Recall that, in general, given a 2-monad S and a pseudo-distributive law of S over T_{coc}, the pseudo-monad for free cocompletion, the category $[(S1)^{op}, Set]$, which is equivalent to $T_{coc}S1$, has the structure of a pseudo-S-algebra (Theorem 7.9). Therefore, an object α of Sk induces a functor of the form

$$\overline{\alpha} : [(S1)^{op}, Set]^k \longrightarrow [(S1)^{op}, Set]$$

given by the composite of

$$[k, [(S1)^{op}, Set]] \xrightarrow{S} [Sk, S[(S1)^{op}, Set]] \xrightarrow{ev_\alpha} S[(S1)^{op}, Set]$$

with the algebra structure

$$S[(S1)^{op}, Set] \longrightarrow [(S1)^{op}, Set]$$

This is a routine extension of the idea that every model of an equational theory supports a semantics for every operation of the theory: $T_{coc}S1$ is a pseudo-S-algebra, so it supports every S-operation; a k-ary S-operation amounts to an object of Sk; and the displayed formula spells out explicitly how such an operation is canonically modelled on a pseudo-S-algebra.

Similarly, but more easily, as $S1$ also possesses an S-algebra structure, an object β of $S2$ yields a functor

$$\beta^\sharp : (S1)^2 \longrightarrow S1$$

This, by composition, yields a functor

$$\beta^\sharp(1, -) : S1 \longrightarrow S1$$

Putting the above altogether we have the following definition:

Definition 9.2 (generic binding signature). For a 2-monad S, a *binding signature* $\Sigma = (O, a)$ is a set of operations O together with an arity function $a : O \longrightarrow Ar_S$ where an element $(k, \alpha, (\alpha_i)_{1 \leq i \leq k})$ of Ar_S consists of a natural number k, an object α of the category Sk, and, for $1 \leq i \leq k$, an object α_i of the category $S2$ together with a strength $X(\alpha_i^\sharp(1, -)) \bullet Y \longrightarrow (X \bullet Y)(\alpha_i^\sharp(1, -))$ over pointed objects Y, i.e., functors Y with a specified element of $Y(1)$

The algebra on a presheaf X associated to an operation of arity $(k, \alpha, (\alpha_i)_{1 \leq i \leq k})$ is

$$\overline{\alpha}(X(\alpha_1^\sharp(1, -), \ldots, \alpha_k^\sharp(1, -))) \longrightarrow X.$$

This definition suffices for our purposes, yielding the level of generality we seek. But obviously, in due course, we should prefer a definition that does not involve the condition at the end: such a definition should be readily obtainable as the condition does not appear explicitly in Fiore *et al*. or Tanaka's work, and it is clear how to avoid it in all the leading examples; but it is not clear yet what is the best condition that implies it in general.

9.2.2 Other possibilities

As a syntactic development in another direction, in [MS03], Miculan and Scagnetto, and also Fiore in [Fio02], gave a typed version of the work in [FPT99]. The category used in the paper [MS03] is $\mathcal{S} = [\mathcal{U}, Set]$, where $\mathcal{U}$ is the category of typed contexts and defined to be the comma category $in \downarrow U$ of the inclusion functor $in : \mathbb{F} \hookrightarrow Set$ and the set U of variable types. Extending the framework presented in this thesis to a version for typed variables is one obvious direction for further study. The construction in [MS03] should involve a pseudo-distributive law for 2-monads on Cat^U.

Another interesting possibility is to fit the direction chosen initially by Gabbay and Pitts in their paper [GP99] and followed by Miculan and others in [GMM03b] into the framework presented in this thesis. They use the presheaf category on $\mathbb{I}$, where $\mathbb{I}$ is the category of natural numbers and injections, and also the notion of Fraenkel-Mostowski set theory, which is equivalent to what is called the Schanuel topos, a particular full subcategory of $[\mathbb{I}, Set]$. These structures are also related to permutation algebras [GMM03a]. Gabbay and Pitts' line of work is considered to be one of the two main directions of category-theoretic research on higher order abstract syntax, both of which coincidentally appeared in the LICS'99 conference [FPT99, GP99, Hof99], but it seems that the two directions can be unified in terms of our framework as the category theoretic construction shown in [GMM03b] seems to fit nicely. Specifically, the Schanuel topos is the free cocompletion on $\mathbb{I}^{op}$ that respects pushouts. So, one could start by replacing *Cat* by the category of small categories with pushouts, and by attempting to emulate the line of argument of this thesis there.

There are other topics where the analysis on pseudo-distributive laws in this thesis can be applied. Investigating such applications is certainly a major direction of further research. One such publicly available is the study of concurrency and bisimulation by Winskel and Cattani [WC04] using open maps and profunctors; the structure used there involves pseudo-comonads and Kleisli constructions. The analysis of pseudo-distributive laws in this thesis can be easily applied to the case of pseudo-comonads. It is also useful to provide a detailed account of the relation between the Kleisli construction and pseudo-distributive laws.

Bibliography

[AC98] R. M. Amadio and P-L Curien. *Domain and Lambda-Calculi*. Cambridge University Press, 1998.

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.

[Bén67] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, number 47 in Lecture Notes in Mathematics, pages 1–77. Springer-Verlag, 1967.

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59:1–41, 1989.

[BLL98] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, 1998.

[Bor94] F. Borceux. *Handbook of Catetorical Algebra I-III*. Cambridge University Press, 1994.

[BW85] M. Barr and C. Wells. *Toposes, Triples and Theories*. Springer-Verlag, 1985.

[BW99] M. Barr and C. Wells. *Category Theory for Computing Science*. Les Publication CRM, 3rd edition, 1999.

[CHP04] Eugenia Cheng, Martin Hyland, and John Power. Pseudo-distributive laws. In Stephen Brookes and Prakash Panangaden, editors, *Electronic Notes in Theoretical Computer Science*, volume 83. Elsevier, 2004.

[CSW97] G. L. Cattani, I. Stark, and G. Winskel. Presheaf models for the π-calculus. In Eugenio Moggi and Giuseppe Rosolini, editors, *Proceedings of 7th International Conference Category Theory and Computer Science, CTCS '97*, volume 1290 of *Lecture Notes in Computer Science*, pages 106–126, Santa Margherita Ligure, Italy, September 1997. Springer.

[Day70] B. Day. On closed categories of functors. In *Lecture Notes in Mathematics 137*, pages 1–38. Springer-Verlag, 1970.

[dB72] N. de Bruijn. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the church-rosser theorem. *Indagationes Mathematicae*, 32:381–391, 1972.

[DPS77] J. Despeyroux, F. Pfenning, and C. Schürmann. Primitive recursion for higher-order abstract syntax. In *Third International Conference on Typed Lambda Calculi and Applications*, volume 1210 of *Lecture Notes in Computer Science*, pages 147–163. Springer-Verlag, April 1977.

[Eil74] S. Eilenberg. *Automata, Languages and Machines*, volume A. Academic Press, 1974.

[FCW99] M. P. Fiore, G. L. Cattani, and G. Winskel. Weak bisimulation and open maps. In Longo [Lon99], pages 67–76.

[Fio02] M. Fiore. Semantic analysis of normalisation by evaluation for typed lambda calculus. In *Proc. PPDP 02*, pages 26–37. ACM Press, 2002.

[FPT99] Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding. In Longo [Lon99], pages 193–202.

[FS95] Peter Freyd and Ross Street. On the size of categories. *Theory and Application of Categories*, 1(9):174–178, 1995.

[FT01] M. Fiore and D. Turi. Semantics of name and value passing. In H. Mairson, editor, *Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science*, pages 93–104, Boston, USA, 2001. IEE Computer Society Press.

[GM96] A. D. Gordon and T. Melham. Five axioms of alpha conversion. In *TPHOLs'96*, volume 1125 of *Lecture Notes in Computer Science*, pages 173–191. Springer-Verlag, 996.

[GMM03a] F. Gadducci, M. Miculan, and U. Montanari. About permutation algebras and sheaves (and named sets, too!). Technical Report UDMI/26/2003/RR, UDMI, 2003.

[GMM03b] F. Gadducci, M. Miculan, and U. Montanari. Some characterization results for permutation algebras. In *Proceedings of COMETA 03*, ENTCS, 2003.

[GP99] M. Gabbay and A. M. Pitts. A new approach abstract syntax involving binders. In Longo [Lon99], pages 214–224.

[GPS95] R. Gordon, A. J. Power, and Ross Street. *Coherence for tricategories*. Number 558 in Memoirs of the American Mathematical Society. American Mathematical Society, 1995.

[Gun92] C. A. Gunter. *Semantics of Programming Languages*. The MIT Press, 1992.

[HH99] A. Hajnal and P. Hamburger. *Set Theory*. Number 48 in London Mathematical Society Student Texts. Cambridge University Press, 1999.

[HH$^+$01] J. Y. Halpern, R. Harper, et al. On the unusual effectiveness of logic in computer science. *The Bulletin of Symbolic Logic*, 7(2):213–236, June 2001.

[HHP93] R. Harper, F. Honsell, and G. Plotkin. A framework for defining logics. *Journal of the ACM*, 40(1):143–184, January 1993.

[HMS01] H. Honsell, M. Miculan, and I. Scagnetto. π-calculus in (co)inductive type theory. *Theoretical Computer Science*, 253(2):239–285, 2001.

[Hof99] M. Hofmann. Semantical analysis of higher-order abstract syntax. In Longo [Lon99], pages 204–213.

[HP00] Martin Hyland and John Power. Symmetric monoidal sketches. In *Proceedings of the 2nd ACM SIGPLAN International Conference on Principles and Practice of Declarative Programming*, pages 280–288, Montreal, Quebec, Canada, 2000. ACM Press.

[IK86] G. B. Im and G. M. Kelly. A universal property of the convolution monoidal structure. *Journal of Pure and Applied Algebra*, 43:75–88, 1986.

[Joy81] A. Joyal. Une théorie combinatoire des séries formelles. *Advances in Mathematics*, 42:1–82, 1981.

[Joy87] A. Joyal. Foncteurs analytiques et espèces de structures. In *Combinatoire énumerative*, volume 1234 of *Lecture Notes in Mathematics*, pages 126–159. Springer-Verlag, 1987.

[Kel64] G. M. Kelly. On maclane's conditions for coherence of natural associativities, commutativities, etc. *Journal of Algebra*, 1:397–402, 1964.

[Kel72a] G. M. Kelly. An abstract approach to coherence. In *Coherence in categories*, volume 281 of *Lecture Notes in Mathematics*, pages 106–147. Springer-Verlag, 1972.

[Kel72b] G. M. Kelly. ”many-variable functorial calculus i”. In *Coherence in categories*, volume 281 of *Lecture Notes in Mathematics*, pages 66–105. Springer-Verlag, 1972.

[Kel74] G. M. Kelly. Coherence theorems for lax algebras and for distributive laws. In *Category Seminar: proceedings, Sydney Category Seminar '72/'73*, volume 420 of *Lecture Notes in Mathematics*, pages 281–375. Springer-Verlag, 1974.

[Kel82] G. Kelly. *The Basic Concepts of Enriched Category Theory*. Cambridge University Press, 1982.

[Kel92] G. M. Kelly. On clubs and data-type constructors. Technical Report 92-1, School of Mathematics and Statistics, The University of Sydney, January 1992.

[Koc70] Anders Kock. Monads on symmetric monoidal closed categories. *Archiv der Mathematik*, 21(1–10), 1970.

[KS74] G. M. Kelly and R. Street. Review of the elements of 2-categories. In *Category Seminar (Proceedings Sydney Category Seminar, 1972-1973)*, number 420 in Lecture Notes in Mathematics, pages 75–103. Springer-Verlag, 1974.

[Lon99] G. Longo, editor. *Proceedings of the Fourteenth Annual Symposium on Logic in Computer Science*. IEEE Computer Science Press, 1999.

[LS86] J. Lambek and P. J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, 1986.

[Mac98] S. MacLane. *Categories for the Working Mathematician*. Springer-verlag, 2nd edition, 1998.

[Mar99] F. Marmolejo. Distributive laws for pseudomonads. *Theory and Applications of Categories*, 5(5):91–147, 1999.

[Mil00] D. Miller. Abstract syntax for variable binders: An overview. In John Lloyd et al, editor, *Computational Logic - CL 2000 First International Conference*, volume 1861 of *Lecture Notes in Artificial Intelligence*, pages 239–253, London, July 2000. Springer-Verlag.

[MM94] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*. Universitext. Springer-Verlag, 1994.

[Mog89] E. Moggi. Computational lambda-calculus and monads. In *LICS 89*, pages 14–23, 1989.

[MS03] M. Miculan and I. Scagnetto. A framework for typed HOAS and semantics. In *Proceedings of the 5th ACM SIGPLAN international conference* *on Principles and Practice of Declarative Programming, PPDP 03*, pages 184–194, Uppsala, Sweden, 2003. ACM Press.

[OT97] P. W. O'Hearn and R. D. Tennent. *Algol-like Languages*. Progress in Theoretical Computer Science Series. Birkhauser, 1997.

[PCW00] John Power, Gian Luca Cattani, and Glynn Winskel. A representation result for free cocompletions. *Journal of Pure and Applied Algebra*, 151(3):273–286, 2000.

[PE88] F. Pfenning and C. Elliot. Higher-order abstract syntax. In *Proceedings of the SIGPLAN'88 Symposium on Language Design and Implementation*, pages 199–208, Atlanta, Georgia, 1988. ACM Press.

[Pit03] A. M. Pitts. Nominal logic, a first order theory of names and binding. *Information and Computation*, 186:165–193, 2003.

[Pow90] A. J. Power. A 2-categorical pasting theorem. *Journal of Algebra*, 129(2):439–445, March 1990.

[Pow99] A. J. Power. Enriched lawvere theories. *Theory and Applications of Categories*, 6(7):83–93, 1999.

[Pow03] J. Power. A unified category-theoretic approach to variable binding. In *Proceedings of the 2003 workshop on Mechanized reasoning about languages with variable binding, MERLIN'03*, pages 1–9. ACM Press, 2003.

[POY04] D. J. Pym, P. W. O'Hearn, and H. Yang. Possible worlds and resources: The semantics of BI. *Theoretical Computer Science*, 315(1):257–305, 2004.

[PT04a] J. Power and M. Tanaka. Pseudo-distributive laws and axiomatics for variable binding. Submitted, 2004.

[PT04b] J. Power and M. Tanaka. A unified category-theoretic approach to substitution in substructural logics. In *Workshop on Logics for Resources, Processes and Programs*, pages 183–192, Turku, Finland, July 2004.

[Pym99] David J. Pym. On bunched predicate logic. In Longo [Lon99], pages 183–192.

[Pym02] D. J. Pym. *The Semantics and Proof Theory of the Logic of the Logic of Bunched Implications*, volume 26 of *Applied Logic Series*. Kluwer Academic Publishers, 2002.

[Pym04] D. J. Pym. Errata and Remarks for *The Semantics and Proof Theory of the Logic of Bunched Implications* [Pym02]. Maintained at: http://www.cs.bath.ac.uk/ pym/BI-monograph-errata.pdf, 2004.

[Sto88] Allen Stoughton. Substitution revisited. *Theoretical Computer Science*, 59(3):317–325, 1988.

[Tan00] M. Tanaka. Abstract syntax and variable binding for linear binders. In *Mathematical Foundations of Computer Science 2000*, volume 1893 of *LNCS*, pages 670–679. Springer, 2000.

[WC04] G. Winskel and G. L. Cattani. Profunctors, open maps and bisimulation. Accepted, 2004.